Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

H. D. Pande, S. B. Misra

Projective Ricci identities in a Finsler space

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.3-4, p. 251–257.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_3-4_251_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Geometria differenziale. — *Projective Ricci identities in a Finsler space*. Nota [(*)] di H. D. PANDE e S. B. MISRA, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Nella presente Nota viene considerato uno spazio di Finsler dotato di un campo tensoriale al quale si applica una connessione sia di Berwald che di Cartan. Ad esse si estendono in vario modo le identità del Ricci.

INTRODUCTION

Let us consider an n-dimensional Finsler space F_n [1] [(1)] equipped with a line element $(x^i, \dot{x}^i)$ $(i = 1, 2, \cdots n)$ and a fundamental metric function $F(x^i, \dot{x}^i)$ which satisfies the conditions invariably imposed upon such a function ([1], Ch. I). The fundamental metric tensor of F_n is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \quad (2) \tag{1.1}$$

which satisfies

$$\dot{\partial}_h g_{ij}(x, \dot{x}) = 2 C_{ijh}(x, \dot{x}) . \tag{1.2}$$

Let us consider a contravariant vector field $X^i(x, \dot{x})$ depending both upon directional as well as positional coordinates x^i. The covariant derivatives of $X^i(x, \dot{x})$ with respect to x^k are given by

$$X^i_{(k)} = \partial_k X^i - \dot{\partial}_m X^i G^m_k + X^m G^i_{mk} \tag{1.3}$$

$$X^i|_k = F \dot{\partial}_k X^i + X^m A^i_{mk} \tag{1.4}$$

$$X^i_{|k} = \partial_k X^i - \dot{\partial}_m X^i G^m_k + X^m \Gamma^{*i}_{mk} \tag{1.5}$$

where $G^i_{mk}(x, \dot{x})$ $\Gamma^{*i}_{mk}(x, \dot{x})$ are the connection coefficients of Berwald and Cartan respectively. And

$$A^i_{kh}(x, \dot{x}) = F(x, \dot{x}) C^i_{kh}(x, \dot{x}) \tag{1.6}$$

$$a) \quad G^i_{hk}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_k G^i_h(x, \dot{x}) , \qquad b) \quad G^i_h(x, \dot{x}) = \dot{\partial}_h G^i(x, \dot{x}) . \tag{1.7}$$

The projective covariant derivative of $X^i(x, \dot{x})$ is given by

$$X^i\|_h = F \dot{\partial}_h X^i + X^m \Pi^i_{mh} \tag{1.8}$$

(*) Pervenuta all'Accademia il 18 settembre 1975.

(1) Numbers in brackets refer to the references at the end of the paper.

(2) $\partial_i \equiv \partial/\partial x^i$ and $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$.

where $\Pi^i_{mh}(x, \dot{x})$ is a projective connection parameter defined by

$$(1.9) \qquad \Pi^i_{hk} \overset{\text{def}}{=} G^i_{hk} - \frac{1}{n+1}(\dot\partial^i_k G^m_{mh} + \dot\partial^i_h G^m_{mk} + \dot{x}^i G^m_{mkh}) .$$

The general commutation formula [3] for tensors of arbitrary rank is given by

$$(1.10) \qquad A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q(h)}\|_k - A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}\|_{k(h)} = A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q(s)} \Pi^s_{hk} -$$

$$-\sum_{\beta=1}^{p} A^{i_1, i_2, \cdots, i_{\beta-1}, s, i_{\beta+1}, \cdots, i_p}_{j_1, j_2, \cdots, j_q} U^{i_\beta}_{shk} + \sum_{\alpha=1}^{q} A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_{\alpha-1}, s, j_{\alpha+1}, \cdots, j_s} U^s_{j_\alpha kh}$$

where

$$(1.11) \qquad U^i_{shk} \overset{\text{def}}{=} \Pi^i_{sh(k)} - FG^i_{shk} .$$

THEOREM 2.1. *The projective Ricci identities for the contravariant and covariant vector fields* $X^i(x, \dot{x})$ *and* $X_i(x, \dot{x})$ *are given by*

$$(2.1\ a) \qquad X^i\|_{[h}\|_{k]} = \dot\partial_{[h} X^i F \|_{k]} + \frac{1}{2} X^s B^i_{shk}$$

and

$$(2.1\ b) \qquad X_i\|_{[h}\|_{k]} \overset{\text{def}}{=} \dot\partial_{[h} X_i F \|_{k]} - \frac{1}{2} X_s B^s_{ihk}$$

where

$$(2.2) \qquad B^i_{shk} \overset{\text{def}}{=} 2 F \dot\partial_{[h} \Pi^i_{k]s} + 2 \Pi^m_{s[h} \Pi^i_{k]m} .$$

Proof. Differentiating (1.8) projectively, we have

$$(2.3) \qquad X^i\|_h\|_k = F\dot\partial_k (X^i\|_h) + X^m\|_h \Pi^i_{mk} + X^i\|_m \Pi^m_{hk} .$$

From equations (1.8) and (2.3), we get

$$(2.4) \qquad X^i\|_h\|_k = F(\dot\partial_k F)(\dot\partial_h X^i) + F^2 \dot\partial^2_{hk} X^i + F(\dot\partial_k X^s)\Pi^i_{sh} + $$
$$+ FX^s(\dot\partial_k \Pi^i_{sh}) + F(\dot\partial_h X^m)\Pi^i_{mk} + X^s \Pi^m_{sh}\Pi^i_{mk} .$$

Similarly, we have

$$(2.5) \qquad X^i\|_k\|_h = F(\dot\partial_h F)(\dot\partial_k X^i) + F^2 \dot\partial^2_{kh} X^i + F(\dot\partial_h X^s)\Pi^i_{sk} + $$
$$+ FX^s(\dot\partial_h \Pi^i_{sk}) + F(\dot\partial_k X^m)\Pi^i_{mh} + X^s \Pi^m_{sk}\Pi^i_{mh} .$$

Subtracting (2.4) and (2.5), we get (2.1). The proof of Theorem 2.1 *b* follows the pattern of the proof of (2.1 *a*) exactly.

THEOREM 2.2. *Let* $A_{ij}(x, \dot{x})$ *be a second order covariant tensor, then the projective Ricci identity for* $A_{ij}(x, \dot{x})$ *is given by*

$$(2.6) \qquad 2 A_{ij}\|_{[h}\|_{k]} = -2 F \|_{[h} \dot\partial_{k]} A_{ij} - A_{is} B^s_{jhk} - A_{sj} B^s_{ihk} .$$

Proof. Let $M^i(x, \dot{x})$ be an arbitrary contravariant vector field such that its inner product with the tensor A_{ij} is given by

$$X_i = A_{ij}(x, \dot{x}) M^j . \tag{2.7}$$

With the help of (2.7) equation (2.3) takes the form

$$\begin{aligned} M^j [A_{ij}\|_h\|_k - A_{ij}\|_k\|_h] + A_{ij}[M^j\|_h\|_k - M^j\|_k\|_h] = (\dot{\partial}_h A_{ij}) M^j F\|_k + \\ + A_{ij}(\dot{\partial}_h M^j) F\|_k - (\dot{\partial}_k A_{ij}) M^j F\|_h - (\dot{\partial}_k M^j) A_{ij} F\|_h - A_{sj} M^j B^s_{ihk} . \end{aligned} \tag{2.8}$$

Using (2.1 *a*) and (2.8), we have

$$M^j [A_{ij}\|_h\|_k - A_{ij}\|_k\|_h + A_{is} B^s_{jhk} + A_{sj} B^s_{ihk} - (\dot{\partial}_h A_{ij}) F\|_k + (\dot{\partial}_k A_{ij}) F\|_h = 0. \tag{2.9}$$

Since $M^j(x, \dot{x})$ is an arbitrary contravariant vector, then (2.6) follows from (2.9).

Similarly, we have

THEOREM 2.3. *Let $A^{ij}(x, \dot{x})$ be a contravariant tensor of order two, then the projective Ricci identity for the tensor $A^{ij}(x, \dot{x})$ is given by*

$$2 A^{ij}\|_{[h}\|_{k]} = 2(\dot{\partial}_{[h} A^{ij}) F\|_{k]} + A^{sj} B^i_{shk} + A^{is} B^j_{shk} . \tag{2.10}$$

THEOREM 2.4. *The projective Ricci identity for a covariant tensor $A_{i_1, i_2, \cdots, i_p}(x, \dot{x})$ of order p is given by*

$$\begin{aligned} A_{i_1, i_2, \cdots, i_p}\|_h\|_k - A_{i_1, i_2, \cdots, i_p}\|_k\|_h = (\dot{\partial}_h A_{i_1, i_2, \cdots, i_p}) F\|_k - \\ - (\dot{\partial}_k A_{i_1, i_2, \cdots, i_p}) F\|_h - \sum_{\beta=1}^{p} A_{i_1, i_2, \cdots, i_{\beta-1}, s, i_{\beta+1}, \cdots, i_p} B^s_{i_\beta hk} . \end{aligned} \tag{2.11}$$

Proof. Let us assume that the theorem holds for a covariant tensor of order m $(m < p)$. Thus, we have

$$\begin{aligned} T_{i_1, i_2, \cdots, i_m}\|_h\|_k - T_{i_1, i_2, \cdots, i_m}\|_k\|_h = (\dot{\partial}_h T_{i_1, i_2, \cdots, i_m}) F\|_k - \\ - (\dot{\partial}_k T_{i_1, i_2, \cdots, i_m}) F\|_h - \sum_{\beta=1}^{m} T_{i_1, i_2, \cdots, i_{\beta-1}, s, i_{\beta+1}, \cdots, i_m} B^s_{i_\beta hk} . \end{aligned} \tag{2.12}$$

Let $A_{i_1, i_2, \cdots, i_m, j}(x, \dot{x})$ be a $(m + 1)$–th order covariant tensor and let $M^i(x, \dot{x})$ as before be an arbitrary contravariant vector field. The inner product of $M^j(x, \dot{x})$ and $A_{i_1, i_2, \cdots, i_m, j}(x, \dot{x})$ is given by

$$T_{i_1, i_2, \cdots, i_m} \overset{\text{def}}{=} A_{i_1, i_2, \cdots, i_m, j}(x, \dot{x}) M^j . \tag{2.13}$$

Substituting the value of $T_{i_1, i_2, i_3, \cdots, i_m}$ from (2.13) in (2.12) we have

$$
\begin{aligned}
(2.14) \qquad & M^j [A_{i_1, i_2, \cdots, i_m, j} \|_h\|_k - A_{i_1, i_2, \cdots, i_m, j}\|_k\|_h] + A_{i_1, i_2, \cdots, i_m j} [M^j\|_h\|_k - \\
& - M^j\|_k\|_h] = (\dot\partial_h M^j) A_{i_1, i_2, \cdots, i_m, j} F \|_k + M^j (\dot\partial_h A_{i_1, i_2, \cdots, i_m, j}) F \|_k - \\
& - (\dot\partial_k M^j) A_{i_1, i_2, \cdots, i_m, j} F \|_h - (\dot\partial_h A_{i_1, i_2, \cdots, i_m, j}) M^j F \|_h - \\
& - \sum_{\beta=1}^{m} A_{i_1, i_2, \cdots, i_{\beta-1}, s, i_{\beta+1}, \cdots, i_m, j} B^s_{i_\beta hk} M^j .
\end{aligned}
$$

Taking into account (2.1 a) and rearranging the terms in (2.14), we have

$$
\begin{aligned}
(2.15) \qquad & M^j [A_{i_1, i_2, \cdots, i_m, j}\|_h\|_k - A_{i_1, i_2, \cdots, i_m, j}\|_k\|_h + A_{i_1, i_2, \cdots, i_m, s} B^s_{j\beta hk} - \\
& - (\dot\partial_h A_{i_1, i_2, \cdots, i_m, j}) F \|_k + (\dot\partial_k A_{i_1, i_2, \cdots, i_m, j}) F \|_h + \\
& + \sum_{\beta=1}^{m} A_{i_1, i_2, \cdots, i_{\beta-1}, s, i_{\beta+1}, \cdots, i_m, j} B^s_{i_\beta hk}] = 0 .
\end{aligned}
$$

Since $M^j (x, \dot x)$ is an arbitrary vector then we get the following equation from (2.15) after replacing the index j by i_{m+1}

$$
\begin{aligned}
(2.16) \qquad & A_{i_1, i_2, \cdots, i_{m+1}}\|_h\|_k - A_{i_1, i_2, \cdots, i_{m+1}}\|_k\|_h = (\dot\partial_h A_{i_1, i_2, \cdots, i_{m+1}}) F \|_k - \\
& - (\dot\partial_k A_{i_1, i_2, \cdots, i_{m+1}}) F \|_h - \sum_{\beta=1}^{m+1} A_{i_1, i_2, \cdots, i_{\beta-1}, s, i_{\beta+1}, \cdots, i_{m+1}} B^s_{i_\beta hk} .
\end{aligned}
$$

Thus, the formula is also true for a covariant tensor of order $(m + 1)$. It has already been established that the formula is true for a covariant tensor of order two. Hence it is true for a covariant tensor of an arbitrary order, say p.

Similarly, we can prove the following:

THEOREM 2.5. *The projective Ricci identity for a contravariant tensor* $A^{i_1, i_2, \cdots, i_q} (x, \dot x)$ *of order* q *is given by*

$$
\begin{aligned}
(2.17) \qquad & A^{i_1, i_2, \cdots, i_q}\|_h\|_k - A^{i_1, i_2, \cdots, i_q}\|_k\|_h = (\dot\partial_h A^{i_1, i_2, \cdots, i_q}) F \|_k - \\
& - (\dot\partial_k A^{i_1, i_2, \cdots, i_q}) F \|_h + \sum_{\beta=1}^{q} A^{i_1, i_2, \cdots, \beta-1, s, i_{\beta+1}, \cdots, i_q} B^s_{i_\beta hk}
\end{aligned}
$$

THEOREM 2.6. *The projective Ricci identity for a mixed tensor of order* $(p + q)$ *(contravariant of order* p *and covariant of order* q*) is given by*

$$
\begin{aligned}
(2.18) \qquad & A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}\|_h\|_k - A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}\|_k\|_h = (\dot\partial_h A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}) F \|_k - \\
& - (\dot\partial_k A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_q}) F \|_k + \sum_{\beta=1}^{p} A^{i_1, i_2, \cdots, i_{\beta-1}, s, i_{\beta+1}, \cdots, i_p}_{j_1, j_2, \cdots, j_q} B^{i_\beta}_{shk} - \\
& - \sum_{\alpha=1}^{q} A^{i_1, i_2, \cdots, i_p}_{j_1, j_2, \cdots, j_{\alpha-1}, s, j_{\alpha+1}, \cdots, j_q} B^s_{j_\alpha hk} .
\end{aligned}
$$

Proof. The proof of Theorem 2.6 follows the pattern of the proofs of Theorems 2.4 and 2.5.

3. THE COMMUTATION FORMULAE IN A CONFORMAL FINSLER SPACE

In a conformal Finsler space [2], we have

$$(3.1)\qquad a)\ \ \bar{g}_{ij}(x,\dot{x}) = e^{2\sigma} g_{ij}(x,\dot{x}), \qquad b)\ \ \overline{F}(x,\dot{x}) = e^{\sigma} F(x,\dot{x})$$

where $\sigma = \sigma(x)$. The following geometric entities have been obtained [2]:

$$(3.2)\qquad a)\ \ \overline{G}^{i}(x,\dot{x}) = G^{i}(x,\dot{x}) - \sigma_m B^{im}(x,\dot{x})$$

$$(3.2)\qquad b)\ \ \overline{G}^{i}_{jk}(x,\dot{x}) = G^{i}_{jk}(x,\dot{x}) - \sigma_m \dot{\partial}_k \dot{\partial}_j B^{im}(x,\dot{x})$$

$$(3.3)\qquad a)\ \ \bar{l}^{i}(x,\dot{x}) = \bar{e}^{\sigma} l^{i}(x,\dot{x}), \qquad b)\ \ \bar{l}_{i}(x,\dot{x}) = e^{\sigma} l_{i}(x,\dot{x})$$

$$(3.4)\qquad \bar{\dot{x}}^{i} = \dot{x}^{i} \quad \text{and} \quad (3.5)\ \ \overline{\Pi}^{i}_{jk}(x,\dot{x}) = \Pi^{i}_{jk}(x,\dot{x})$$

$$(3.6)\qquad a)\ \ \overline{C}_{ijk}(x,\dot{x}) = e^{2\sigma} C_{ijk}(x,\dot{x}) \quad \text{and} \quad b)\ \ \overline{C}^{i}_{jk}(x,\dot{x}) = C^{i}_{jk}(x,\dot{x})$$

$$(3.7)\qquad \overline{A}^{i}_{jk}(x,\dot{x}) = e^{\sigma} A^{i}_{jk}(x,\dot{x})$$

where we have

$$(3.8)\qquad \sigma_m \stackrel{\text{def}}{=} \partial_m \sigma,\ B^{ij}(x,\dot{x}) \stackrel{\text{def}}{=} \frac{1}{2} F^2 g^{ij} - \dot{x}^i \dot{x}^j,\ l^i(x,\dot{x}) \stackrel{\text{def}}{=} \frac{\dot{x}^i}{F(x,\dot{x})},$$
$$l_i(x,\dot{x}) \stackrel{\text{def}}{=} g_{ik} l^k(x,\dot{x}),\ A^{i}_{mj}(x,\dot{x}) \stackrel{\text{def}}{=} F(x,\dot{x})\, C^{i}_{mj}(x,\dot{x}).$$

THEOREM 3.1. *When* $F_n(x,\dot{x})$ *and* $\overline{F}_n(x,\dot{x})$ *are in conformal correspondence, we have*

$$(3.9\,a)\qquad \overline{C}^{m}_{ij(\bar{h})}\overline{\|}_k - \overline{C}^{m}_{ij}\overline{\|}_{k(\bar{h})} = [C^{m}_{ij(h)}\|_k - C^{m}_{ij}\|_{k(h)}] +$$
$$+ [C^{*m}_{ijs}\Pi^{s}_{hk} + C^{s}_{ij} L^{*m}_{skh} - C^{m}_{sj} L^{*s}_{ikh} - C^{m}_{is} L^{*s}_{jkh}] +$$
$$+ (e^{\sigma} - 1) F [C^{m}_{sj} G^{s}_{ikh} + C^{m}_{is} G^{s}_{jkh} - C^{s}_{ij} G^{m}_{skh}]$$

$$(3.9\,b)\qquad \overline{A}^{i}_{jm(\bar{h})}\overline{\|}_k - \overline{A}^{i}_{jm}\overline{\|}_{k(\bar{h})} = e^{\sigma} [\{A^{i}_{jm(h)}\|_k - A^{i}_{jm}\|_{k(h)}\} +$$
$$+ \{A^{s}_{jm} L^{*i}_{skh} - A^{i}_{sm} L^{*s}_{jkh} - A^{i}_{js} L^{*s}_{mkh} + \Pi^{s}_{hk} A^{*i}_{jms}\} +$$
$$+ (1 - e^{\sigma}) F \{A^{s}_{jm} G^{i}_{skh} - A^{i}_{sm} G^{s}_{jkh} - A^{i}_{js} G^{s}_{mkh}\}].$$

where

$$(3.10\,a)\qquad C^{*m}_{ijs} \stackrel{\text{def}}{=} \sigma_l [(\dot{\partial}_q C^{m}_{ij}) B^{ql}_{s} - C^{q}_{ij} B^{ml}_{kq} + C^{m}_{qj} B^{ql}_{si} + C^{m}_{iq} B^{ql}_{sj}]$$

$$(3.10\,b)\qquad L^{*m}_{skh} \stackrel{\text{def}}{=} \sigma_l [(\dot{\partial}_q \Pi^{m}_{sk}) B^{lq}_{h} - \Pi^{q}_{sk} B^{ml}_{hq} + \Pi^{m}_{qk} B^{ql}_{hs} + \Pi^{m}_{sq} B^{ql}_{hk} + \partial^{\sigma} F B^{ml}_{hks}]$$

and

$$(3.10\,c)\qquad B^{ml}_{hks} \stackrel{\text{def}}{=} \dot{\partial}_h \dot{\partial}_k \dot{\partial}_s B^{ml}.$$

Proof. The commutation formula (1.10) for $\overline{C}^m_{ij}$ in conformal Finsler space is given by

$$\overline{C}^m_{ij(\overline{h})}\overline{\|}_k - \overline{C}^m_{ij}\overline{\|}_{k(\overline{h})} = \overline{C}^m_{ij(\overline{s})}\overline{\Pi}^s_{hk} + \overline{C}^s_{ij}\overline{U}^m_{skh} - \overline{C}^m_{sj}\overline{U}^s_{ikh} - \overline{C}^m_{is}\overline{U}^s_{jkh}\,. \tag{3.11}$$

From (1.3) we obtain

$$\overline{C}^m_{ij(\overline{s})} = \dot{\partial}_s\overline{C}^m_{ij} - (\dot{\partial}_q\overline{C}^m_{ij})\overline{G}^q_s + \overline{C}^q_{ij}\overline{G}^m_{qk} - \overline{C}^m_{qj}\overline{G}^q_{is} - \overline{C}^m_{iq}\overline{G}^q_{js} \tag{3.12}$$

and

$$\overline{U}^i_{shk} = \overline{\Pi}^i_{sh(\overline{k})} - \overline{F}\overline{G}^i_{shk}\,. \tag{3.13}$$

Using equations (3.12), (3.13), (3.1 *b*), (3.2 *b*), (3.5), (3.6 *b*) and arranging the terms in (3.11), we have

$$\begin{aligned}\overline{C}^m_{ij(\overline{h})}\overline{\|}_k - \overline{C}^m_{ij}\overline{\|}_{k(\overline{h})} = {}& C^m_{ij(s)}\Pi^s_{hk} + C^s_{ij}\Pi^m_{sk(h)} - C^m_{sj}\Pi^s_{ik(h)} - \\ & - C^m_{is}\Pi^s_{jk(h)} + C^{*m}_{ijs}\Pi^s_{kh} + C^s_{ij}L^{*m}_{skh} - C^m_{sj}L^{*s}_{ikh} - \\ & - C^m_{is}L^{*s}_{jkh} + e^\sigma F(C^m_{sj}G^s_{ikh} + C^m_{is}G^s_{jkh} - C^s_{ij}G^m_{skh})\,.\end{aligned} \tag{3.14}$$

Adding and subtracting the terms $FC^m_{sj}G^s_{ikh}$, $FC^m_{is}G^s_{jkh}$ and $FC^s_{ij}G^m_{skh}$ in (3.14) we get (3.9 *a*).

Similarly, we can prove (3.9 *b*).

THEOREM 3.2. *When* $F_n(x,\dot{x})$ and $\overline{F}_n(x,\dot{x})$ *are in conformal correspondence, we have*

$$\begin{aligned}\overline{G}^i_{(\overline{h})}\overline{\|}_k - \overline{G}^i\overline{\|}_{k(\overline{h})} = {}& [\{G^i_{(h)}\|_k - G^i\|_{k(h)}\} + \sigma_l\{B^{sl}B^{*i}_{skh} - \\ & - B^{sl}\Pi^i_{sk(h)} - B^{il}_{(s)}\Pi^s_{hk} + Q^{*i}_{mls}\Pi^s_{hk}\} + FG^i_{skh}\{G^s(1-e^\sigma) + \\ & + \sigma_l B^{sl}e^\sigma\}] + G^s L^i_{skh}\end{aligned} \tag{3.15 a}$$

$$\begin{aligned}\overline{G}^i_{j(\overline{h})}\overline{\|}_k - \overline{G}^i_j\overline{\|}_{k(\overline{h})} = {}& [\{G^i_{j(h)}\|_k - G^i_j\|_{k(h)}\} + \sigma_l\{G^m_j L^i_{mkh} + \\ & + G^i_m L^m_{jkh} - B^{ml}_j\Pi^i_{mk(h)} - B^{il}_m\Pi^m_{jk(h)} - B^{il}_{j(m)}\Pi^m_{hk} + \\ & + B^{ml}_j B^{*i}_{mkh} + B^{il}_m B^{*m}_{jkh} + P^{*il}_{jm}\Pi^m_{hk}\} + FG^i_{mkh}\{G^m_j(1-e^\sigma) + \\ & + \sigma_l e^\sigma B^{ml}_j\} + FG^m_{jkh}\{G^i_m(1-e^\sigma) + \sigma_l e^\sigma B^{il}_m\}]\,.\end{aligned} \tag{3.15 b}$$

where

$$L^{*i}_{mkh} \overset{\text{def}}{=} \sigma_l L^i_{mkh} \tag{3.16 a}$$

$$B^{*i}_{skh} \overset{\text{def}}{=} \sigma_l\{\dot{\partial}_m\Pi^i_{sk})B^{ml}_h + B^{il}_{mh}\Pi^m_{sk} - \Pi^i_{mk}B^{ml}_{sh} - \Pi^i_{sm}B^{ml}_{kh} - e^\sigma FB^{il}_{skh}\} \tag{3.16 b}$$

$$Q^{*i}_{mls} \overset{\text{def}}{=} \{(\dot{\partial}_m G^i)B^{ml}_s + B^{im}G^l_{ms} - \sigma_l B^{il}B^{ml}_s - G^m B^{il}_{ms} + \sigma_l B^{ml}B^{il}_{ms}\}. \tag{3.16 c}$$

$$\begin{aligned}P^{*il}_{jm} \overset{\text{def}}{=} {}& \{(\dot{\partial}_q G^i_j)B^{ql}_m - (\dot{\partial}_q B^{il}_j)B^{ql}_m - G^q_j B^{il}_{qm} + \\ & + \sigma_l B^{ql}_j B^{il}_{qm} - G^i_q B^{ql}_{jm} - \sigma_l B^{il}_q B^{ql}_{jm} + B^{iq}_j G^l_{lm}\}\,.\end{aligned} \tag{3.16 d}$$

THEOREM 3.3. *When* $F_n(x, \dot{x})$ and $\overline{F}_n(x, \dot{x})$ *are in conformal correspondence, the commutation formula for* $\overline{G}^i_{jl}(x, \dot{x})$ *is given by*

$$(3.17)\qquad \begin{aligned} &\overline{G}^i_{jl(\overline{h})}\overline{\|}_k - \overline{G}^i_{jl}\overline{\|}_{k(\overline{h})} = [\{G^i_{jl(h)}\|_k - G^i_{jl}\|_{k(h)}\} + \\ &+ \sigma_q \{(G^s_{jl} L^i_{skh} - G^i_{sl} L^s_{jkh} - B^{iq}_{lj(s)} \Pi^s_{hk} - G^i_{js} L^s_{lkh} - \\ &- B^{sq}_{lj} \Pi^i_{sk(h)} + B^{iq}_{ls} \Pi^s_{jk(h)} + B^{iq}_{sj} \Pi^s_{lk(h)} - B^{sq}_{lj} \overset{*}{L}{}^i_{skh} + \\ &+ B^{iq}_{ls} \overset{*}{L}{}^s_{jkh} + B^{iq}_{sj} \overset{*}{L}{}^s_{lkh} + P^{iq}_{jls} \Pi^s_{hk}) + e^{\sigma} F (B^{sq}_{lj} G^i_{skh} - \\ &- B^{iq}_{ls} G^s_{jkh} - B^{iq}_{sj} G^s_{lkh}) + B^{im}_{lj} G^q_{ms} \Pi^s_{hk}\} + \\ &+ F(e^{\sigma} - 1)\{G^i_{js} G^s_{lkh} + G^i_{sl} G^s_{jkh} - G^s_{jl} G^j_{skh}\}] . \end{aligned}$$

Proof. The pattern of the proofs of Theorems 3.2 and 3.3 is the same as that of Theorem 3.1.

REFERENCES

[1] H. RUND (1959) – *The Differential Geometry of Finsler space*, Springer Verlag.
[2] H. D. PANDE (1974) – *A study of some problems in Finsler space–Pustakasthan*, Gorakhpur (India).
[3] H. D. PANDE and S. B. MISRA – *The commutation formulae involving Berwald's and projective covariant derivative* (Comm.).