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Operators of $A - -p$ type

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Operatori funzionali. — *Operators of* $A - p$ *type*. Nota (*) di B. E. Rhoades, presentata dal Socio G. Sansone.

Riassunto. — Questa Nota estende i risultati di Gh. Constantin [1], e K. Iséki [2] ad una classe di spazi più ampia.

Let X, Y be normed linear spaces, $\alpha_n(T) = \inf\{\|T - B\| : B \in \mathscr{R}_n(X, Y)\}$, where $\mathscr{R}_n(X, Y) = \{B : B : X \to Y \text{ and } \dim R(B) \leq n\}$. We shall define

$$(1) \qquad |A, p| = \left\{x : \left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} |a_{nk} x_k|\right)^p\right)^{1/p} < \infty, 0 < p < \infty\right\} \cdot$$

$$\cdot |A, \infty| = \left\{x : \sup_{n \geq 0}\left\{\sum_{k=0}^{\infty} |a_{nk} x_k|\right\} < \infty\right\}.$$

An operator $T \in B(X, Y)$ shall be called of $A - p$ type if $\{\alpha_n(T)\}_{n=0}^{\infty}$ is an element of the corresponding $A - p$ space. With $A = C$, the Cesàro matrix of order one, this definition reduces to that of [1]. For A a regular weighted mean method with decreasing weights, this definition reduces to that of [2]. Pietsch [3] calls an operator $T \in B(X, Y)$ of l^p-type if $\sum_{n=0}^{\infty} (\alpha_n(T))^p$ is finite.

We shall restrict our attention to those matrices A which satisfy the condition

$$(2) \qquad |a_{n,2k}| + |a_{n,2k+1}| \leq M |a_{nk}| \qquad \text{for each } k \text{ and } n,$$

where M is a constant independent of n and k.

Theorem 1. *For each fixed* A *satisfying* (2) *and each* p, *the set of operators of* $A - p$ *type is a linear space.*

Proof. From [3, p. 249], $\alpha_n(S)$ is monotone decreasing in n, and $\alpha_{m+n}(S + T) \leq \alpha_m(S) + \alpha_n(T)$ for each integer pair $m, n \geq 0$.

$$\left(\sum_{k=0}^{\infty} |a_{nk} \alpha_k(S + T)|\right)^p = \left(\sum_{k=0}^{\infty} |a_{n,2k}| \alpha_{2k}(S + T) + \sum_{k=0}^{\infty} |a_{n,2k+1}| \alpha_{2k}(S + T)\right)^p \leq$$

$$\leq \left(\sum_{k=0}^{\infty} (|a_{n,2k}| + |a_{n,2k+1}|) \alpha_{2k}(S + T)\right)^p \leq$$

$$\leq \left(M \sum_{k=0}^{\infty} |a_{nk}| (\alpha_k(S) + \alpha_k(T))\right)^p.$$

(*) Pervenuta all'Accademia il 27 settembre 1975.

For $p \geq 1$ the result follows from Minkowski's inequality. For $0 < p < 1$ use the facts that, for any real positive numbers a, b, c, d, $(a+b)^p \leq a^p + b^p$, and $(c+d)^{1/p} \leq \tau (c^{1/p} + d^{1/p})$, where $\tau = \max\{2^{1/p-1}, 1\}$. The case for $p = \infty$ is trivial.

PROPOSITION 1. *Let* T *be an* A — p *type operator,* A *a matrix satisfying* $\sum_{k=0}^{n} |a_{nk}| \geq \delta > 0$ *for all* n. *Then* T *is an* l^p *type operator.*

Proof.

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} |a_{nk}| \alpha_k(T) \right)^p \geq \sum_{n=0}^{\infty} \left(\alpha_n(T) \sum_{k=0}^{n} |a_{nk}| \right)^p \geq \delta^p \sum_{n=0}^{\infty} (\alpha_n(T))^p.$$

Let $|A|$ denote the matrix with entries $|a_{nk}|$.

PROPOSITION 2. *Let* $|A| \in B(l^p)$. *Then* $l^p \subseteq |A, p|$.

Proof. Simply observe that, if $x \in l^p$, then, from (1), $|A| \in B(l^p)$ is equivalent to $x \in |A, p|$.

COROLLARY 1. *Every* l^p *type operator is an* A — p *type operator.*

COROLLARY 2. *If* A *satisfies* $\{|a_{n0}|\} \notin l^p$, *then the only* A — p *type operator is zero.*

Proof. Suppose $T \neq 0$. Then $\alpha_0(T) = \|T\| \neq 0$.

$$\sum_{n=0}^{N} \left(\sum_{k=0}^{\infty} |a_{nk} \alpha_k(T)| \right)^p \geq \sum_{n=0}^{N} |a_{n0} \alpha_0(T)|^p \to +\infty \quad \text{as } N \to \infty.$$

If, in addition to (2), we add the restriction that each column of A contains at least one nonzero entry, and if we define $\beta_{A,p} : A - p \to R$ by

$$\beta_{A,p}(T) = \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} |a_{nk} \alpha_k(T)| \right)^p \right)^{1/p},$$

then $\beta_{A,p}$ satisfies the following properties:

1) $\beta_{A,p}(T) \geq 0$,

2) $\beta_{A,p}(T) = 0$ implies $T = 0$,

3) $\beta_{A,p}(\lambda T) = |\lambda| \beta_{A,p}(T)$ for all scalars λ, and

4) there exists a number $\sigma_p \geq 1$ such that

$$\beta_{A,p}(S+T) \leq \sigma_p [\beta_{A,p}(S) + \beta_{A,p}(T)].$$

PROPOSITION 3. *Let* A *satisfy the condition of Proposition 1,* $\{T_n\}$ *a fundamental sequence of operators of* A — p *type such that there exists a* $T \in B(X, Y)$ *with* $\lim_n T_n x = Tx$ *for all* $x \in X$. *Then* T *is an operator of* A — p *type and* $\beta_{A,p} - \lim T_n = T$.

Proof. From Proposition 1, $\|S\| \leq (1/\delta)\, \beta_{A,p}(S)$ for every S of $A-p$ type, so that $\{T_n\}$ is a fundamental sequence in B (X, Y). Since $\lim_n T_n x = TX$ for each $x \in X$, $\{T_n\}$ converges to T in B (X, Y).

From [3], $|\alpha_n(S) - \alpha_n(T)| \leq \|S - T\|$ for each $S, T \in B(X, Y)$ and each $n \geq 0$. Thus $|\alpha_s(T - T_n) - \alpha_s(T_m - T_n)| \leq \|(T - T_n) - (T_m - T_n)\| = \|T - T_m\|$, which yields $\lim_m \alpha_s(T_m - T_n) = \alpha_s(T - T_n)$.

For each $\varepsilon > 0$ there exists a number $n_0(\varepsilon)$ such that for $n, m \geq n_0(\varepsilon)$,

$$\beta_{A,p}(T_m - T_n) = \left(\sum_{r=0}^{\infty}\left(\sum_{k=0}^{\infty} |a_{rk}|\, \alpha_k(T_m - T_n)\right)^p\right)^{1/p} \leq \varepsilon .$$

Taking the limit as $m \to \infty$ we obtain

$$\beta_{A,p}(T - T_n) = \left(\sum_{r=0}^{\infty}\left(\sum_{k=0}^{\infty} |a_{rk}|\, \alpha_k(T - T_n)\right)^p\right)^{1/p}, \qquad n \geq n_0(\varepsilon) .$$

Thus $T - T_n$ is of $A-p$ type, which implies T is of $A-p$ type, and $\beta_{A,p} - \lim_n T_n = T$.

PROPOSITION 4. *Let* X *be a normed space,* Y *a Banach space,* A *satisfy* $\sup_k \sum_{n=0}^{\infty} |a_{nk}|^p < \infty$. *Then the space of operators of* $A-p$ *type is complete.*

Proof. Let T be a finite rank operator of rank q. Then

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} |a_{nk}|\, \alpha_k(T)\right)^p = \sum_{n=0}^{q}\left(\sum_{k=0}^{\infty} |\alpha_{nk}|\, \alpha_k(T)\right)^p + \sum_{n=q+1}^{\infty}\left(\sum_{k=0}^{\infty} |a_{nk}|\, \alpha_k(T)\right)^p .$$

For $n > q$,

$$\left(\sum_{k=0}^{\infty} |a_{nk}|\, \alpha_k(T)\right)^p = \left(\sum_{k=0}^{q} |a_{nk}|\, \alpha_k(T)\right)^p \leq \max_{0 \leq k \leq q} |a_{nk}|^p \left(\sum_{k=0}^{q} a_k(T)\right)^p \cdot$$

$$\cdot \sum_{n=q+1}^{\infty} \max_{0 \leq k \leq q} |a_{nk}|^p \leq \sup_k \sum_{n=0}^{\infty} |a_{nk}|^p < \infty .$$

Therefore every finite rank operator is an operator of $A-p$ type.

PROPOSITION 5. *Let* X, Y, Z *be normed linear spaces. If* $T \in B(X, Y)$, *and* $S \in B(Y, Z)$ *is of* $A-p$ *type, then* ST *is of* $A-p$ *type and* $\beta_{A,p}(ST) \leq \|T\|\, \beta_{A,p}(S)$. *Also, if* $T \in B(X, Y)$ *is of* $A-p$ *type and* $S \in B(Y, Z)$, *then* ST *is of* $A-p$ *type and* $\beta_{A,p}(ST) \leq \|S\|\, \beta_{A,p}(T)$.

PROPOSITION 6. *For each fixed* A *and* p, *the set of operators of* $A-p$ *type is a completely* $\beta_{A,p}$*-normed ideal.*

PROPOSITION 7. *An operator* $T \in B(m)$ *of the form* $Tx = \{a_i x_i\}$, $\{a_i\}$ *a monotone decreasing bounded sequence, is an operator of* A — *p type if and only if*

$$\beta_{A,p}(T) = \left\{ \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} | a_{nk} a_k | \right)^p \right\}^{1/p} < \infty .$$

COMMENTS.

1) It is an open question whether condition (2) is necessary for Theorem 1 to be true.

2) The condition on A in Proposition 1 cannot be removed. To see this, fix $p > 0$ and choose α so that $\alpha > 1 + 1/p$, and define A by $a_{nk} = (n+1)^{-\alpha}$ for $k \leq n$, $a_{nk} = 0$ for $k > n$. The identity operator is not of l^p type, but

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} | a_{nk} \alpha_k (I) | \right)^p \leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (n+1)^{-\alpha} \right)^p = \sum_{n=0}^{\infty} (n+1)^{-p} \alpha < \infty ,$$

and I is of A — p type.

3) For any matrix A satisfying (2) and the hypotheses of Proposition 1 and 2, the operators of A — p type are precisely those of l^p type. This is a very interesting result in light of the fact that the sequences spaces l^p are properly contained in $|A, p|$ for many choices of A.

4) The example of Comment 2 shows that there exist matrices A satisfying (2) and $|A| \in B(l^p)$ and such that the corresponding class of A — p type operators properly contains the l^p type operators.

5) For any fixed $p > 0$, the collection of all matrices A satisfying (2) and the hypotheses of Propositions 1 and 2 gives rise to an infinite collection of quasi-norms $\beta_{A,p}$, all of which are equivalent to the quasi-norm $\left(\sum_{n=0}^{\infty} (\alpha_n(T))^p \right)^{1/p}$, which is a norm for $p \geq 1$.

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