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# Some applications of the fundamental characterization theorem of R. C. Bose to partial geometries

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**Geometrie finite.** — *Some applications of the fundamental characterization theorem of R. C. Bose to partial geometries.* Nota di JOSEPH A. THAS e FRANK DE CLERCK, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Ad ogni assegnata geometria parziale ne viene associata un'altra (che può dirsi ad essa complementare). Vengono poi caratterizzate le strutture d'incidenza ottenibili a partire da un piano $\pi$ proiettivo (non necessariamente desarguesiano) d'ordine $q$ col sopprimere da $\pi$ i punti di un $\{qd - q + d ; d\}$-arco, $d$ essendo un intero soddisfacente alle $1 < d < q$.

## 1. INTRODUCTION

1.1. A (finite) partial geometry $(r, k, t)$ is an incidence structure $S = (P, B, I)$ with a symmetric incidence relation satisfying the following axioms:

(i) each point is incident with $r$ lines ($r \geq 2$) and two distinct points are incident with at most one line;

(ii) each line is incident with $k$ points ($k \geq 2$) and two distinct lines are incident with at most one point;

(iii) if $x$ is a point and L is a line not incident with $x$, then there are exactly $t$ ($t \geq 1$) points $x_1, x_2, \cdots, x_t$ and $t$ lines $L_1, L_2, \cdots, L_t$ such that $x I L_i I x_i I L$, $i = 1, 2, \cdots, t$.

If $|P| = v$ and $|B| = b$, then $v = k((k-1)(r-1)+t)/t$ and $b = r((r-1)(k-1)+t)/t$ [3]. Consequently $t \mid k(k-1)(r-1)$ and $t \mid r(r-1)(k-1)$. We also remark that $t \leq k$ and $t \leq r$.

1.2. If the points $x, y$ (resp. lines L, M) of S are collinear (resp. concurrent), then we write $x \sim y$ (resp. $L \sim M$); otherwise we write $x \nsim y$ (resp. $L \nsim M$).

The graph G of the partial geometry $S = (P, B, I)$ is the graph $(P, E)$, where $E = \{\{x, y\} \subset P \,\|\, x \sim y\}$. It is easy to prove that G is strongly regular with parameters $v = k((k-1)(r-1)+t)/t$, $n_1 = r(k-1)$, $p^1_{11} = (r-1)(t-1)+k-2$, $p^2_{11} = rt$ (1) [3]. The graph G of a partial geometry $(r, k, t)$ is called a geometric graph $(r, k, t)$.

A strongly regular graph G is defined to be pseudo geometric $(r, k, t)$ if its parameters $v, n_1, p^1_{11}, p^2_{11}$ are given by (1), where $r, k, t$ are integers with $r \geq 2$, $k \geq 2$, $1 \leq t \leq k$, $1 \leq t \leq r$. In [2] R.C. Bose establishes a sufficient condition for a pseudo geometric graph $(r, k, t)$ to be geometric $(r, k, t)$: the pseudo geometric graph $(r, k, t)$ is geometric $(r, k, t)$ if

$$k > \tfrac{1}{2}(r(r-1) + t(r+1)(r^2 - 2r + 2)) . \tag{2}$$

(*) Nella seduta dell'11 giugno 1975.

## 1.3. EXAMPLES OF PARTIAL GEOMETRIES

(*a*) The balanced incomplete block designs with $\lambda = 1$ are the partial geometries $(r, k, k)$ [3].

(*b*) The nets of degree $r\,(\geq 2)$ and order $k\,(\geq 2)$ are the partial geometries $(r, k, r-1)$ [3].

(*c*) The partial geometries for which $t = 1$ are the generalized quadrangles [3].

(*d*) In a finite projective plane of order $q$, any non-void set of $l$ points may be described as a $\{l; d\}$-arc, where $d\,(d \neq 0)$ is the greatest number of collinear points in the set. For given $q$ and $d\ (d \neq 0)$, $l$ can never exceed $(d-1)(q+1)+1$, and an arc with that number of points will be called a maximal arc [1]. Equivalently, a maximal arc may be defined as a non-void set of points meeting every line in just $d$ points or in none at all. It is not difficult to prove that a necessary condition for the existence of a maximal arc (as a proper subset of a given plane) is that $d$ should be a factor of $q$ [1]. But the condition is not sufficient; J. A. Thas [10] has proved that, in the desarguesian plane of order $q = 3^h\ (h > 1)$, there is no $\{2q+3; 3\}$-arc. In [6] R. H. F. Denniston proves that the condition does suffice in the case of any desarguesian plane of order $2^h$.

Let K be a $\{qd - q + d; d\}$-arc, $1 < d < q$, of a projective plane $\pi$ (not necessarily desarguesian) of order $q$. Define points of the partial geometry S as the points of $\pi$ which are not contained in K. Lines of S are the lines of $\pi$ which are incident with $d$ points of K. The incidence is that of $\pi$. Now it is easy to prove that the configuration S so defined is a partial geometry with parameters $(q - q/d + 1, q - d + 1, q - q/d - d + 1)$ ([8], [9], [11]). Moreover, if $\pi$ is desarguesian then it is also possible to construct partial geometries $(qd - q + d, q, d - 1)$ and $(q(q-d+1)/d, q, (q-d)/d)$ ([8], [9]).

## 2. MAIN THEOREM

2.1. THEOREM. *If there exists a partial geometry* $S = (P, B, I)$ *with parameters*

$$r = u + 1,\quad k = s + 1,\ t \qquad \textit{for which}\quad \gamma = s - t > 0,\quad t \mid us,$$

$u\gamma \leq (\gamma + 1)(s - \gamma)$ *and* $u(2s - \gamma(\gamma+2)(\gamma^2+1)) > (s-\gamma)(\gamma(\gamma+1) - 2)$, *then there also exists a partial geometry* $\bar{S} = (\bar{P}, \bar{B}, \bar{I})$ *with parameters* $\bar{r} = s + 1 - t$, $\bar{k} = (su + t)/t$, $\bar{t} = u(s - t)/t$.

*Proof.* The graph $G = (P, E)$ of S has parameters

$$v = (s+1)(su+t)/t, \qquad n_1 = (u+1)s, \qquad p^1_{11} = (s-1) + u(t-1),$$

$$p^2_{11} = (u+1)t \qquad (u \geq 1, s \geq 1, 1 \leq t \leq u+1, 1 \leq t < s).$$

The complementary graph $\overline{G} = (P, \overline{E})$, $\overline{E} = \{\{x, y\} \subset P \| x \nsim y\}$, of G is strongly regular with parameters $\bar{v} = v = (s+1)(su+t)/t$, $\bar{n}_1 = n_2 = su(s+1-t)/t$, $\bar{p}^1_{11} = p^2_{22} = su(s+1-t)/t - (u+1)(s-t) - 1$, $\bar{p}^2_{11} = \bar{p}^1_{22} = u(s+1-t)(s-t)/t$. Let $\bar{k} = (su+t)/t$, $\bar{r} = s+1-t$ and $\bar{t} = u(s-t)/t$.

Then $\bar{r}, \bar{k}, \bar{t}$ are integers with $\bar{k} \geq 2, \bar{r} \geq 2, 1 \leq \bar{t} < \bar{k}, 1 \leq \bar{t} \leq \bar{r}$ (this follows from $\gamma = s - t > 0$, $t \mid us$ and $u\gamma \leq (\gamma+1)(s-\gamma)$). Now it is easy to check that $\overline{G}$ is pseudo geometric $(\bar{r}, \bar{k}, \bar{t})$. Since $u(2s - \gamma(\gamma+2)(\gamma^2+1)) > (s-\gamma)(\gamma(\gamma+1)-2)$, it follows from (2) that $\overline{G}$ is geometric $(\bar{r}, \bar{k}, \bar{t})$. Consequently there exists a partial geometry $(\bar{r}, \bar{k}, \bar{t})$.

*Remark.* $\overline{P} = P$, the elements of $\overline{B}$ are the grand cliques of the graph $\overline{G}$, and $\overline{I}$ is the natural incidence relation [2].

2.2. *Corollaries.*

(*a*) By applying this theorem to nets, we obtain the well-known theorem of Bruck-Shrikhande [5].

(*b*) If there exists a partial geometry $(q - q/d + 1, q - d + 1, q - q/d - d + 1)$, with $1 < d < q$ and $2q > \delta^4 - \delta^3 + \delta^2 + \delta - 2$ $(q = \delta d)$, then there exists a partial geometry $(q/d, q+1, q/d)$ and consequently a balanced incomplete block design with parameters $k^* = q/d$, $r^* = q+1$, $\lambda^* = 1$.

## 3. Embedding of the complement of a maximal arc in a projective plane

3.1. *Ovoids and spreads.*

Let $S = (P, B, I)$ be a partial geometry $(u+1, s+1, t)$. If V is a set of points (resp. lines) of S no two of which are collinear (resp. concurrent), then it is easy to prove that $|V| \leq (su+t)/t$. If $|V| = (su+t)/t$, then V is called an ovoid (resp. spread) of S. A necessary condition for the existence of an ovoid (resp. spread) is that $t$ should be a factor of $su$.

3.2. *The complement of a maximal arc.*

Let K be a $\{qd - q + d; d\}$-arc, $1 < d < q$, of a projective plane $\pi$ of order $q$ (not necessarily desarguesian). If we delete the points of K from $\pi$, then the incidence structure of the remaining points and lines has the following properties:

(i) there are two types of lines: lines of type (I) are incident with $q+1$ points and lines of type (II) are incident with $q+1-d$ points $(1 < d < q)$;

(ii) each point is incident with $q/d$ lines of type (I) and $q+1-q/d$ lines of type (II);

(iii) any two distinct points are both incident with exactly one line.

Conversely let D be an incidence structure with the above properties. We may ask the question whether or not it is possible to embed D in a projective plane of order $q$ by suitably extending the lines of type (II).

3.3. THEOREM. *If* $D = (P, B, I)$ *is an incidence structure with the above properties* (i)-(iii) *and if* $2q > d^4 - d^3 + d^2 + d - 2$, *then* D *is embeddable in a projective plane* $\pi$ *of order* $q$ *by suitably extending the lines of type* (II). *Moreover the set of the "new" points is a* $\{qd - q + d; d\}$*-arc of that plane.*

*Proof.* Suppose that L is a line of type (I) and that $M \in B - \{L\}$. We shall prove that L and M are concurrent. Let $x$ I M and $x \not{I}$ L. From (iii) follows that there are exactly $q + 1$ lines incident with $x$ and concurrent with L. Moreover $x$ is incident with exactly $q + 1$ lines (see (ii)). Consequently L and M are concurrent.

Let $D_1 = (P, B_1, I_1)$, where $B_1$ is the set of lines of type (I) and where $I_1$ is induced by the incidence relation I. From the previous remark follows easily that $D_1$ is a partial geometry $(q/d, q + 1, q/d)$.

Let $D_2 = (P, B_2, I_2)$, where $B_2$ is the set of lines of type (II) and where $I_2$ is induced by the incidence relation I. Suppose that $L \in B_2$ and $x \not{I}$ L. There are exactly $q + 1 - d$ lines of B incident with $x$ and concurrent with L. Among these lines are the $q/d$ lines of $B_1$ which are incident with $x$. Consequently $B_2$ contains exactly $q + 1 - d - q/d$ lines which are incident with $x$ and concurrent with L. Hence $D_2$ is a partial geometry $(q + 1 - q/d, q + 1 - d, q + 1 - d - q/d)$.

Now we consider the geometry $D_2^* = (B_2, P, I_2)$, which is a partial geometry $(q + 1 - d, q + 1 - q/d, q + 1 - d - q/d)$. From $1 < q/d < q$ and $2q > d^4 - d^3 + d^2 + d - 2$ there follows that the incidence structure $D_3 = (B_2, P_1, I_3)$, with $P_1$ the set of grand cliques of the complement $\overline{G}_2^*$ of the graph of $D_2^*$ and with $I_3$ the natural incidence relation, is a partial geometry $(d, q + 1, d)$ (see 2.2.b.). We remark that each element of $P_1$ is incident with $q + 1 = ((q - q/d)(q - d) + (q + 1 - d - q/d))/(q + 1 - d - q/d)$ elements of $B_2$, and consequently the grand cliques of $\overline{G}_2^*$ are the spreads of $D_2$ (i.e. the ovoids of $D_2^*$).

Let us consider the incidence structure $D' = (P \cup P_1, B_1 \cup B_2, I \cup I_3)$. First of all we remark that $|P \cup P_1| = |P| + |P_1| = (q + 1)(q - d + 1) + dq - q + d = q^2 + q + 1$. If $x, y \in P$, $x \neq y$, then there is exactly one element of $B_1 \cup B_2$ which is incident with $x$ and $y$ (the element L defined by $x$ I L I $y$); there is exactly one element of $B_1 \cup B_2$ which is incident with two given elements $x, y$, $x \in P$ and $y \in P_1$ (the element L of the spread $y$ of $D_2$ for which $x$ I L); finally there is exactly one element of $B_1 \cup B_2$ which is incident with two given elements $x, y \in P_1$, $x \neq y$ (since $D_3$ is a partial geometry $(d, q + 1, d)$, the spreads $x, y$ of $D_2$ have exactly one line in common). Consequently any two distinct points of $D'$ are both incident with exactly one line of $D'$. Since each line of $D'$ is incident with $q + 1$ points of $D'$ (remark that each element of $B_2$ is incident with $d$ elements of $P_1$), we conclude that D′ is a $2-(q^2+q+1; q+1, 1)$ design, i.e. a projective plane of order $q$. Evidently $P_1$ is a $\{qd-q+d; d\}$-arc of the plane D′.

*Remarks.* (*a*) For $d=2$ we have the theorem of Bose-Shrikhande [4] about the embedding of the complement of a complete oval in a projective plane of even order.

(*b*) An analogous reasoning yields the following theorem: Suppose that S is a partial geometry $(q+1-q/d, q+1-d, q+1-d-q/d)$, $1<d<q$, for which the following axioms are satisfied:

(i) $2q > \delta^4-\delta^3+\delta^2+\delta-2$, where $q=\delta d$;

(ii) S has a family V of spreads such that any two non concurrent lines of S are contained in exactly one element of V.

Then S is a partial geometry arising from a $\{qd-q+d; d\}$-arc in a projective plane of order $q$.

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