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Joseph A. Thas

Flocks of non-singular ruled quadrics in PG (3,q)

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Geometrie finite. — *Flocks of non-singular ruled quadrics in* PG (3, q). Nota di JOSEPH A. THAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Se Q è una quadrica rigata non singolare di $S_{3,q}$, $q+1$ coniche non degeneri tracciate su Q diconsi costituire un *fascio* (flock) quando esse ricoprono Q completamente, il che val quanto dire che tali coniche risultano a due a due prive di punti comuni. Qui si dimostra che, mentre per *q pari* ossia potenza di 2) *ogni* fascio risulta *lineare* (e cioè formato dalle sezioni di Q coi piani passanti per una retta priva di punti a comune con Q), quando q è *dispari* esistono sempre dei fasci *non lineari*.

1. INTRODUCTION

An ovoid O of the threedimensional projective space PG (3, q), $q > 2$, is a set of q^2+1 points no three of which are collinear. The circles of O are the sets $P \cap O$, where P is a plane of PG (3, q) with $|O \cap P| > 1$. A flock of O is a set F of mutually disjoint circles such that, with the exception of precisely two points, every point of O is on a (necessarily unique) circle of F.

If L is a line of PG (3, q) which has no point in common with O, then the circles $P \cap O$, where P is a plane containing L with $|P \cap O| > 1$, constitute a so-called linear flock of O. That each flock of O is linear was proved by the Author in the even case [2] and by W.F. Orr in the odd case [1].

This paper deals with the flocks of a non-singular ruled quadric Q of PG (3, q). First of all we remark that the circles of the quadric Q are by definition the irreducible conics on Q. A flock of Q is a set F of $q+1$ mutually disjoint circles of Q. If L is a line of PG (3, q) which has no point in common with Q, then the circles $P \cap Q$, where P is a plane containing L, constitute a so-called linear flock of Q.

2. THEOREM. *Each flock of the non-singular ruled quadric* Q *of* PG (3, q), *q even, is linear.*

Proof. Suppose that $F = \{C_1, C_2, \cdots, C_{q+1}\}$ is a flock of the non-singular ruled quadric Q. The nucleus of the circle C_i is denoted by n_i.

We shall prove that $L = \{n_1, n_2, \cdots, n_{q+1}\}$ is a line. For that purpose it is sufficient to show that every plane of PG (3, q) has at least one point in common with L [2].

a) Let P be the tangent plane of Q at $x \in Q$. Through x passes a circle of the flock, say C_i. The tangent line of C_i at x is contained in P, and so $n_i \in P$.

b) Let P be a plane for which $P \cap Q \in F$. If $C_i = P \cap Q$, then $n_i \in P$.

(*) Nella seduta dell'11 giugno 1975.

c) Finally let P be a plane for which $P \cap Q = C$ is an irreducible conic which is not contained in F. Then $|C_i \cap C| \in \{0, 1, 2\}$. As $q+1$ is odd there exists a C_j such that $|C_j \cap C| = 1$. There follows that C_j and C have a common tangent line T at their common point. So $n_j \in T \subset P$, from which $n_j \in P$.

Hence every plane of PG $(3, q)$ has at least one point in common with L. Consequently L is a line of PG $(3, q)$. Next we remark that the polar planes of $n_1, n_2, \cdots, n_{q+1}$, with respect to the symplectic polarity π defined by Q, are the planes of the circles $C_1, C_2, \cdots, C_{q+1}$. As L is a line, these $q+1$ planes all pass through the polar line of L with respect to π. We conclude that the flock F is linear.

3. THEOREM. *Each non-singular ruled quadric* Q *of* PG $(3, q)$, q *odd, has a non-linear flock.*

Proof. We shall use a technique which is due to W. F. Orr [1].

a) Without loss of generality we assume that Q is represented by the equation $x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0$. If $y(y_0, y_1, y_2, y_3) \in Q$ and if P_y is the polar plane of y with respect to Q, then the circle $Q \cap P_y$ is denoted by C_y or $C_y(y_0, y_1, y_2, y_3)$. For any two points $y(y_0, y_1, y_2, y_3)$ and $z(z_0, z_1, z_2, z_3)$, we pose $y \cdot z = y_0 z_0 + y_1 z_1 - y_2 z_2 - y_3 z_3$, $\|y\| = y \cdot y$ ($\|y\| = 0 \Longleftrightarrow y \in Q$), $y \times z = (y \cdot z)^2 - \|y\| \|z\|$.

b) Consider two distinct circles $C_y(y_0, y_1, y_2, y_3)$ and $C_z(z_0, z_1, z_2, z_3)$. The common points of the line yz and Q are determined by the equation $(y_0 + hz_0)^2 + (y_1 + hz_1)^2 - (y_2 + hz_2)^2 - (y_3 + hz_3)^2 = 0$ or $\|z\| h^2 + 2(y \cdot z) h + \|y\| = 0$. The discriminant of this equation is $4(y \times z)$.

Consequently we have:

$$|C_y \cap C_z| = 2 \Longleftrightarrow |yz \cap Q| = 2 \Longleftrightarrow y \times z \text{ is a nonzero square in GF}(q);$$

$$|C_y \cap C_z| = 1 \Longleftrightarrow |yz \cap Q| = 1 \Longleftrightarrow y \times z = 0;$$

$$|C_y \cap C_z| = 0 \Longleftrightarrow |yz \cap Q| = 0 \Longleftrightarrow y \times z \text{ is a nonsquare in GF}(q).$$

c) We write $C_y \sim C_z$ if and only if the circles C_y and C_z have a common tangent circle; otherwise we write $C_y \nsim C_z$.

Suppose that $C_y \sim C_z$. Then there exists a circle C_u for which $y \times u = z \times u = 0$. Hence $(y \cdot u)^2 = \|y\| \|u\|$ and $(z \cdot u)^2 = \|z\| \|u\|$. Consequently $\|y\| \|z\| \|u\|^2 = (y \cdot u)^2 (z \cdot u)^2$, and so $\|y\| \|z\|$ is a square in GF (q).

Conversely, consider two circles C_y and C_z for which $\|y\| \|z\|$ is a square in GF (q). If $C_y = C_z$, then C_y and C_z have a common tangent circle. So we suppose that $C_y \neq C_z$. Let $v(v_0, v_1, v_2, v_3)$ be a point of $C_y - C_z$. Then we have $v \cdot y = 0$ and $v \cdot z \neq 0$. If $t(y_0 + hv_0, y_1 + hv_1, y_2 + hv_2, y_3 + hv_3)$, $h \neq 0$, then $C_t \cap C_y = \{v\}$. We remark that $\|t\| = \|y\| + 2h(y \cdot v) + h^2 \|v\| = \|y\|$. Now there holds $t \times z = ((y \cdot z) + h(v \cdot z))^2 - \|y\| \|z\| = (v \cdot z)^2 h^2 + 2(y \cdot z)(v \cdot z) h + (y \cdot z)^2 - \|y\| \|z\|$. As the discriminant $4(y \cdot z)^2 (v \cdot z)^2 - 4(y \cdot z)^2 (v \cdot z)^2 +$

$+ 4 \|y\| \|z\| (v \cdot z)^2$ is a nonzero square in GF (q), there is at least one circle C_t for which $t \times z = 0$ or $|C_t \cap C_z| = 1$. Hence $C_y \sim C_z$.

From the preceding follows that $C_y \sim C_z$ if and only if $\|y\| \|z\|$ is a square in GF (q). Consequently the relation $\sim$ is an equivalence relation, separating the circles of Q into two equivalence classes.

d) Suppose that the circles C_y and C_z are disjoint and that $C_y \sim C_z$ (it is easy to show geometrically that such circles always exist). Now we consider the $q + 1$ circles C_v which are orthogonal to C_y and C_z (remark that the circles C_v are mutually disjoint). There holds $v \cdot y = v \cdot z = 0$ and consequently $v \times y = - \|v\| \|y\|$, $v \times z = - \|v\| \|z\|$. Hence $(v \times y)(v \times z) = \|v\|^2 \|y\| \|z\|$ is a nonzero square in GF (q), and so $v \times y$ and $v \times z$ are both nonzero squares or both nonsquares in GF (q). There follows that $|C_v \cap C_y| = |C_v \cap C_z| = 2$ or $|C_v \cap C_y| = |C_v \cap C_z| = 0$. The set of the $(q + 1)/2$ circles C_v for which $|C_v \cap C_y| = |C_v \cap C_z| = 2$ (resp. $|C_v \cap C_y| = |C_v \cap C_z| = 0$) is denoted by V (resp. V′).

If $C_{v_1}, C_{v_2} \in V$ (resp. $C_{v_1}, C_{v_2} \in V'$), then $- \|v_1\| \|y\|$ and $- \|v_2\| \|y\|$ are both nonzero squares (resp. nonsquares) and so $C_{v_1} \sim C_{v_2}$. If $C_{v_1} \in V$ and $C_{v_2} \in V'$, then $- \|v_1\| \|y\|$ is a nonzero square and $- \|v_2\| \|y\|$ is a nonsquare, and so in this case $C_{v_1} \nsim C_{v_2}$. Consequently all elements of V belong to one equivalence class of $\sim$, and all elements of V′ belong to the other equivalence class of $\sim$.

Next we consider the circles C_w which are orthogonal to each circle C_v. From the preceding follows that for each circle C_w we have $|C_w \cap C_v| = 2 \; \forall C_v \in V$ (resp. V′), or $|C_w \cap C_v| = 0 \; \forall C_v \in V$ (resp. V′). The set of the $(q + 1)/2$ circles C_w for which $|C_w \cap C_v| = 2 \; \forall C_v \in V$, or $|C_w \cap C_v| = 0 \; \forall C_v \in V'$, is denoted by W′; the set of the $(q + 1)/2$ circles C_w for which $|C_w \cap C_v| = 0 \; \forall C_v \in V$, or $|C_w \cap C_v| = 2 \; \forall C_v \in V'$, is denoted by W. Evidently $V \cup W = F$ and $V' \cup W' = F'$ are non-linear flocks of Q, and so the theorem is completely proved.

REFERENCES

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