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Decomposition of pseudo-projective tensor fields in a second order recurrent Finsier space

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Geometria differenziale. — *Decomposition of pseudo–projective tensor fields in a second order recurrent Finsler space.* Nota [(*)] di AWDHESH KUMAR, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Dato in uno spazio di Finsler un campo tensoriale pseudo–proiettivo (secondo la definizione di B. B. Sinha e di S. P. Singh) si mostra la possibilità di decomposizione del tipo dato in altri più semplici.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n [1] [(1)] equipped with with $2n$ line elements $(x^i, \dot{x}^i)$ and a fundamental metric function $F(x, \dot{x})$ which is positively homogeneous of degree one in its directional arguments. The metric tensor of the space is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \qquad \dot{\partial}_i \equiv \partial/\partial \dot{x}^i. \tag{1.1}$$

The covariant derivative of a vector field $X^i(x, \dot{x})$ with respect to x^k in the sense of Berwald is given by [(2)]

$$X^i_{(k)} = \partial_k X^i - (\dot{\partial}_h X^i) G^h_k + X^h G^i_{hk}, \tag{1.2}$$

where $G^i(x, \dot{x})$ are Berwald's connection coefficients positively homogeneous of degree two in $\dot{x}^i$, s.

The projective covariant derivative [5] of $X^i(x, \dot{x})$ with respect to x^k is given by

$$X^i_{((k))} = \partial_k X^i - (\dot{\partial}_m X^i) \Pi^m_{rk} \dot{x}^r + X^h \Pi^i_{hk}, \tag{1.3}$$

where

$$\Pi^i_{hk}(x, \dot{x}) \overset{\text{def}}{=} \left\{ G^i_{hk} - \frac{1}{(n+1)} (2\, \dot{\partial}^i_{(h} G^r_{k)r} + \dot{x}^i G^r_{rkh}) \right\} \tag{1.4}$$

are projective connection coefficients positively homogeneous of degree zero in x^i, s and also satisfy the following relations:

$$a)\ \ \Pi^i_{hkr} \dot{x}^h = 0 \quad , \quad b)\ \ \Pi^i_{hk} \dot{x}^h = \Pi^i_k. \tag{1.5}$$

(*) Pervenuta all'Accademia il 3 luglio 1975.

(1) The numbers in square brackets refer to the references given in the end of the paper.

(2) $2\, A_{(hk)} = A_{hk} + A_{kh}$ and $2\, A_{[hk]} = A_{hk} - A_{kh}$.

The commutation formulae involving the projective covariant derivative are given by

$$(\dot\partial_k T^i_j)_{((h))} - \dot\partial_k T^i_{j((h))} = T^i_r \Pi^r_{jkh} - T^r_j \Pi^i_{rkh} \tag{1.6}$$

and

$$2\, T^i_{j[((h))((k))]} = -(\dot\partial_r T^i_j)\, Q^r_{hk} - T^i_r\, Q^r_{jhk} + T^r_j\, Q^i_{rhk}\,, \tag{1.7}$$

where

$$Q^i_{hjk}(x, \dot x) \overset{\text{def}}{=} 2\{\partial_{[k} \Pi^i_{j]h} - \Pi^i_{rh[j} \Pi^r_{k]} + \Pi^r_{h[j} \Pi^i_{k]r}\}. \tag{1.8}$$

Two more tensor fields $H^i_j(x, \dot x)$ and $W^i_j(x, \dot x)$ are respectively given by

$$H^i_j(x, \dot x) = 2\, \partial_j G^i - (\partial_h \dot\partial_j G^i)\, \dot x^h + 2\, G^i_{jl} G^l - (\dot\partial_l G^i)\, \dot\partial_j G^l \tag{1.9}$$

and

$$W^i_j(x, \dot x) = H^i_j - H\delta^i_j - \frac{1}{(n+1)} (\dot\partial_r H^r_j - \dot\partial_j H)\, \dot x^i, \tag{1.10}$$

which satisfy the following relations:

(1.11) $a)$ $H^i_{hj} = \frac{2}{3} \dot\partial_{[h} H^i_{j]}$, $b)$ $H^i_{hjk} = \dot\partial_h H^i_{jk}$, $c)$ $H^i_{hk} \dot x^h = H^i_k$ and

$d)$ $H^i_{hjk} \dot x^h = H^i_{jk}$

(1.12) $a)$ $W^i_{hjk} \dot x^h = W^i_{jk}$, $b)$ $W^i_{hk} \dot x^h = W^i_k$, $c)$ $W^i_{hj} = \frac{2}{3} \dot\partial_{[h} W^i_{j]}$ and

$d)$ $\dot\partial_h W^i_{jk} = W^i_{hjk}$.

Pseudo-projective tensor fields $W^{i*}_j(x, \dot x)$ are defined by Sinha and Singh [6] as

$$W^{*i}_j(x, \dot x) = aW^i_j + bH^i_j\,, \tag{1.13}$$

where a and b are scalar functions of $(x, \dot x)$ and homogeneous of degree zero in $\dot x^i$. The pseudo-projective tensor fields satisfy the (3) following identities [6]:

(1.14) $a)$ $W^{*i}_{hk} = W^{*i}_{jhk} \dot x^j$, $b)$ $W^{*i}_{jhk} = \dot\partial_j W^{*i}_{hk}$,

$$W^{*i}_{[lhj]} = 0\,, \tag{1.15}$$

$$W^*_{ikhj} + W^*_{jhkl} = \frac{2}{3} [g_{ik} \dot\partial^2_{l[h} W^{*i}_{j]} + g_{il} \dot\partial^2_{k[j} W^{*i}_{h]}]\,, \tag{1.16}$$

$$W^*_{lkhj} + W^*_{jhkl} = \frac{2}{3} [g_{ik} \dot\partial^2_{l[h} W^{*i}_{j]} + g_{ih} \dot\partial^2_{j[k} W^{*i}_{l]}] \tag{1.17}$$

(3) $3\, A_{[hkr]} = A_{hkr} + A_{krh} + A_{rhk}$.

and

(1.18) $$W^{*}_{lkhj} - W^{*}_{ljhk} + W^{*}_{hjlk} - W^{*}_{hklj} = \frac{2}{3}\,[g_{i[k}\,\dot\partial^{2}_{j]l}\,W^{*i}_{h} + g_{i[k}\,\dot\partial^{2}_{j]h}\,W^{*i}_{l}],$$

where

$$W^{*}_{lkhj}(x,\dot x) \overset{\text{def}}{=} g_{ik}\,W^{*i}_{lhj}\,.$$

The first and second order pseudo-projective recurrent tensor fields are given by

(1.19) $$W^{*i}_{jkh((s))} = \lambda_s\,W^{*i}_{jkh}$$

and

(1.20) $$W^{*i}_{jkh((s))((m))} = a_{sm}\,W^{*i}_{jkh}\,,$$

where λ_s and a_{sm} are recurrence vector and tensor fields respectively. Transvecting (1.19) and (1.20) by $\dot x^{j}$ and noting equation (1.14 a) and the fact $\dot x^{i}_{((k))} = 0$, we get

(1.21) $$W^{*i}_{kh((s))} = \lambda_s\,W^{*i}_{kh}$$

and

(1.22) $$W^{*i}_{kh((s))((m))} = a_{sm}\,W^{*i}_{kh}\,.$$

The following relation between recurrence vector and tensor fields is also satisfied as

(1.23) $$a_{sm} = \lambda_{s((m))} + \lambda_s\,\lambda_m\,.$$

2. DECOMPOSITION OF PSEUDO-PROJECTIVE TENSOR FIELDS

We consider the decomposition of pseudo-projective tensor field $W^{*i}_{kh}(x,\dot x)$ as follows:

(2.1) $$W^{*i}_{kh}(x,\dot x) = \dot x^{i}\,\varphi_{kh}\,,$$

where $\varphi_{kh}(x,\dot x)$ is any non zero homogeneous tensor field of first order in its directional arguments and $\dot x^{i}\,\lambda_i = \rho$ (Constant). Differentiating (2.1) partially with respect to $\dot x^{j}$ and noting equation (1.14 b), we obtain

(2.2) $$W^{*i}_{jkh} = \dot x^{i}\,\varphi_{jkh} \qquad \text{for} \quad i \neq j\,,$$

where

(2.3) $$\varphi_{jkh}(x,\dot x) \overset{\text{def}}{=} \dot\partial_j\,\varphi_{kh}\,.$$

Here now we suppose that the above decomposition conditions which hold for pseudo-projective tensor fields, are also satisfied by the projective entities $Q^{i}_{jkh}(x,\dot x)$.

THEOREM 2.1. *In an n-dimensional Finsler space, the decomposition tensor field* $\varphi_{jkh}(x, \dot{x})$ *satisfies the following identities:*

$$\varphi_{[lhj]} = 0, \tag{2.4}$$

$$\dot{x}^i (g_{ik} \varphi_{lhj} + g_{il} \varphi_{kjh}) = \frac{2}{3} [g_{ik} \dot{\partial}^2_{l[h} W^{*i}_{j]} + g_{il} \dot{\partial}^2_{k]j} W^{*i}_{h]}], \tag{2.5}$$

$$\dot{x}^i (g_{ik} \varphi_{lhj} + g_{ih} \varphi_{jkl}) = \frac{2}{3} [g_{ik} \dot{\partial}^2_{l[h} W^{*i}_{j]} + g_{ih} \dot{\partial}^2_{j[k} W^{*i}_{l]}] \tag{2.6}$$

and

$$\dot{x}^i (g_{ik} \varphi_{[lh]j} - g_{ij} \varphi_{[lh]k}) = \frac{1}{3} [g_{j[k} \dot{\partial}^2_{j]l} W^{*i}_{h} + g_{i[k} \dot{\partial}^2_{j]h} W^{*i}_{l}]. \tag{2.7}$$

Proof. In view of the decomposition defined by equation (2.2), the identities (1.15), (1.16), (1.17) and (1.18) yield respectively the required results (2.4), (2.5), (2.6) and (2.7).

THEOREM 2.2 *In an* F_n *the decomposition tensor fields* $\varphi_{jkh}(x, \dot{x})$ *and* $\varphi_{kh}(x, \dot{x})$ *satisfy the second order recurrency condition.*

Proof. Differentiating (2.2) twice projectively with respect to x^s and x^m, we get

$$W^{*i}_{jkh((s))((m))} = \dot{x}^i \varphi_{jkh((s))((m))}, \tag{2.8}$$

which in view of equations (1.22) and (2.2) reduces to

$$a_{sm} \varphi_{jkh} = \varphi_{jkh((s))((m))}. \tag{2.9}$$

Transvecting (2.9) by $\dot{x}^j$ and noting the homogeneity property of the decomposition tensor field $\varphi_{jkh}(x, \dot{x})$, we obtain

$$a_{sm} \varphi_{kh} = \varphi_{kh((s))((m))} \tag{2.10}$$

which proves the theorem.

THEOREM 2.3. *In an* F_n, *if* λ_s *is independent of* $\dot{x}^i$, *the recurrence tensor field* a_{sm} *satisfies*

$$(\dot{\partial}_j a_{sm} - \dot{\partial}_m a_{sj}) = 0. \tag{2.11}$$

Proof. In view of (2.3), differentiating partially (2.10) with respect to $\dot{x}^j$, we get

$$(\dot{\partial}_j a_{sm}) \varphi_{kh} + a_{sm} \varphi_{jkh} = \dot{\partial}_j \{(\varphi_{kh((s))})_{((m))}\}, \tag{2.12}$$

which in view of equation (1.23), commutation formula (1.6) and the fact that λ_s is independent of $\dot{x}^i$ reduces to

$$(\dot{\partial}_j (a_{sm}) \varphi_{kh} = - \{ \varphi_{rh(s))} \Pi^r_{kmj} + \varphi_{kr((s))} \Pi^r_{hmj} + \varphi_{kh((r))} \Pi^r_{smj} \}. \tag{2.13}$$

Interchanging the indices m and j and substracting the result thus obtained from it, we get the required result (2.11).

THEOREM 2.4. *In an* F_n *if the recurrence vector* λ_s *is independent of direction, the equation*

$$\{(\dot{\partial}_j a_{sm}) \varphi_{kh} - (\dot{\partial}_k a_{sm}) \varphi_{jh}\} \dot{x}^m = 0 \tag{2.14}$$

holds.

Proof. Commutating (2.13) with respect to the indices j and k we obtain

$$\{(\dot{\partial}_j a_{sm}) \varphi_{kh} - (\dot{\partial}_k a_{sm}) \varphi_{jh}\} = 2\{\Pi^r_{hm[k} \varphi_{j]r((s))} + \Pi^r_{sm[k} \varphi_{j]h((r))}\}. \tag{2.15}$$

Transvecting (2.15) by $\dot{x}^m$ and using equation (1.5a), we get the required result.

THEOREM 2.5. *In an n-dimensional Finsler space, the skew symmetric part of the recurrence tensor field* $a_{sm}(x, \dot{x})$ *satisfies the first order recurrence condition under the decomposition defined by* (2.1) *and* (2.2).

Proof. In view of commutation formula (1.7) commutating (2.10) with respect to the indices s and m we get

$$(a_{sm} - a_{ms}) \varphi_{kh} = -(\dot{\partial}_r \varphi_{kh}) Q^r_{sm} - \varphi_{rh} Q^r_{ksm} - \varphi_{kr} Q^r_{hms}, \tag{2.16}$$

which in view of equations (2.1), (2.2) and (2.3) reduces to

$$(a_{sm} - a_{ms}) \varphi_{kh} = -\{\varphi_{rkh} \varphi_{sm} + \varphi_{rh} \varphi_{ksm} + \varphi_{kr} \varphi_{hms}\} \dot{x}^r. \tag{2.17}$$

Differentiating the above equation projective covariantly with respect to x^j and using the first order recurrence condition of the recurrence tensor fields $\varphi_{jkh}(x, \dot{x})$ and φ_{kh}, we obtain

$$\begin{aligned}\{(a_{sm} - a_{ms})_{((j))} + \lambda_j (a_{sm} - a_{ms})\} \varphi_{kh} &= \\ = -2\lambda_j \{\varphi_{rkh} \varphi_{sm} + \varphi_{rh} \varphi_{ksm} + \varphi_{kr} \varphi_{hms}\} \dot{x}^r&.\end{aligned} \tag{2.18}$$

In view of (2.17), the above equation reduces to

$$(a_{sm} - a_{ms})_{((j))} = \lambda_j (a_{sm} - a_{ms}). \tag{2.19}$$

which proves Theorem 2.5.

THEOREM 2.6. *Under the decomposition* (2.1) *and* (2.2), *if is independent of* $\dot{x}^i$ *the relation*

$$\dot{x}^r \{\varphi_{rkh} \varphi_{[mj} + \varphi_{rh} \varphi_{k[mj} + \varphi_{kr} \varphi_{h[mj}\} \lambda_{s]} = 0. \tag{2.20}$$

Proof. Differentiating (2.10) projective covariantly with respect to x^j and using the first order recurrence property of $\varphi_{kh}(x, \dot{x})$, we get

$$(a_{sm((j))} + \lambda_j a_{sm}) \varphi_{kh} = \varphi_{kh((s))((m))((j))}. \tag{2.21}$$

In view of commutation formula (1.7) commutating the above equation with respect to the indices m and j, we get

$$(2.22) \qquad \varphi_{kh}\{a_{sm((j))} - a_{sj((m))} + \lambda_j a_{sm} - \lambda_m a_{sj}\} = = -\{\dot{\partial}_r \varphi_{kh((s))} Q^r_{mj} + \varphi_{rh(s))} Q^r_{kmj} + \varphi_{kr((s))} Q^r_{hmj} + \varphi_{kh((r))} Q^r_{smj}\}.$$

In view of (2.1), (2.2), (2.3) and the fact that λ_s is independent of $\dot{x}^i$, the above equation reduces to

$$(2.23) \qquad \varphi_{kh}(a_{sm(j))} - a_{sj((m))} + \lambda_j a_{sm} - \lambda_m a_{sj}) = -\dot{x}^r\{\lambda_s(\varphi_{rkh}\varphi_{mj} + + \varphi_{rh}\varphi_{kmj} + \varphi_{kr}\varphi_{hmj}) + \lambda_r \varphi_{kh}\varphi_{smj}\}.$$

Interchanging cyclically the indices s, m and j in (2.23) two more similar result obtained. Adding all the three equations in view of the identity (2.4) the symmetric property of the recurrence tensor field a_{sm} i.e ($a_{sm} = a_{ms}$), we obtain the theorem.

THEOREM 2.7. *In an* F_n *the recurrence tensor field* $\varphi_{kh}(x, \dot{x})$ *satisfies the relation*

$$(2.24) \qquad \varphi_{kh[((s))((m))((j))]} = 0.$$

Proof. Commutating (2.21) with respect to the indices m and j, we get

$$(2.25) \qquad \varphi_{kh}(b_{s[m((j))]} + b_{s[m}\lambda_{j]}) = \varphi_{kh((s))[((m))((j))]}.$$

Interchanging cyclically the indices s, m and j in the above equation two more equations are obtained. Adding all the two equations thus obtained with (2.25) in view of the symmetric property of the tensor field $a_{sm}(x, \dot{x})$, we get the required result (2.24).

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