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Andrzej Lasota, Franciszek Hugon Szafraniec

On Hammerstein integral equations

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Equazioni integrali. — *On Hammerstein integral equations.* Nota (*) di ANDRZEJ LASOTA e FRANCISZEK HUGON SZAFRANIEC, presentata dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si usa il teorema dell'applicazione aperta alla teoria delle equazioni integrali non lineari del tipo di Hammerstein. Si mostra che dall'unicità delle soluzioni per un insieme aperto di nuclei segue l'esistenza delle soluzioni stesse.

It is known [1] that for some boundary value problems for nonlinear differential equations the uniqueness of solutions implies the existence. In this note we discuss this phenomenon for Hammerstein integral equations and present a new method of obtaining solutions of these equations. An essential topological tool used here is Schauder's theorem on invariance of domains. For simplicity sake we describe our idea in the classical L^2-spaces rather than in an abstract form of operator equations.

1. Let (S, μ) be a measure space. Let $f : S \times R \to R$ be a given function satisfying Carathéodory conditions (i.e. $f(s, x)$ is measurable in s and continuous in x) and such that for each $x \in L^2(S)$ the function $f(\cdot, x(\cdot))$ is in $L^2(S)$. Let $\mathcal{K} \subset L^2(S \times S)$ be a given set of kernels. Then the integral operator

$$x \to \int_S K(\cdot, s) f(s, x(s)) \mu(ds), \qquad K \in \mathcal{K} \tag{1}$$

maps $L^2(S)$ into itself.

THEOREM. *Given f and $\mathcal{K}$ assume that for each $h \in L^2(S)$ and $K \in \mathcal{K}$ the integral equation*

$$x(t) = \int_S K(s, t) f(s, x(s)) \mu(ds) + h(t), \qquad t \in S \tag{2}$$

admits at most one solution. Assume, moreover, that $\mathcal{K}$ is open in $L^2(S \times S)$. Then for each $h \in L^2(S)$ equation (2) has exactly one solution.

Proof. Fix K in $\mathcal{K}$ and consider the vector field I—T, where I denotes the identity mapping in $L^2(S)$ and T the integral operator given by (1). It is well known that I—T is a continuous compact vector field. Moreover, the uniqueness of the solutions of equation (2) implies that I—T is a one to one mapping. Thus by the Schauder theorem on invariance of domains the range of I—T is open. We claim that $(I-T)(L^2(S))$ is closed. Suppose the contrary. Then there exists a sequence x_n in $L^2(S)$ such that $x_n - T(x_n)$ is a Cauchy

(*) Pervenuta all'Accademia il 7 luglio 1975.

sequence and x_n is not. In other words for every $\varepsilon > 0$ there are x_p and x_q such that

$$\| x_p - x_q - (T(x_p) - T(x_q)) \| < \varepsilon \tag{3}$$

and

$$\| x_p - x_q \| \geq \delta \tag{4}$$

where δ is a positive constant independent of ε.

Set $x_0 = x_p - x_q - T(x_p) + T(x_q)$ and denote by y_0 the function $f(\cdot, x_p(\cdot)) - f(\cdot, x_q(\cdot))$. Observe that $\| y_0 \| > 0$ provided $\varepsilon < \delta$. To see this notice that (3) and (4) imply $\delta \leq \| x_0 \| \leq \| x_0 - T(x_p) + T(x_q) \| + \| T(x_p) - - T(x_q) \| \leq \varepsilon + \| K \| \, \| y_0 \|$ and consequently $\| y_0 \| \geq (\delta - \varepsilon) \| K \|^{-1}$ (here recall that $\| K \|$ is the L^2-norm of the kernel K). Now choose ε such that the ball of center K and radius $\varepsilon \| y_0 \|^{-1}$ is contained in $\mathscr{K}$ and then fix x_p and x_q. Set $\tilde{K}(s, t) = y_0(s_0) x_0(t) \| y_0 \|^{-2}$. Since $\| \tilde{K} \| = \| x_0 \| \, \| y_0 \|^{-1}$ the function $\tilde{K}$ is in $L^2(S \times S)$ and $K + \tilde{K}$ belongs to $\mathscr{K}$. On the other hand

$$\int_S \tilde{K}(s, t) y_0(s) \, \mu(ds) = \| y_0 \|^{-2} \int_S x_0(t) y_0^2(s) \, \mu(ds) = x_0(t), \quad t \in S.$$

This implies that

$$x_p - (T + \tilde{T})(x_p) = x_q - (T + \tilde{T})(x_q)$$

where $\tilde{T}$ is the integral operator corresponding to $\tilde{K}$ due to (1). The above equality together with (4) shows that there are two distinct solution x_p and x_q of equation (2) with the same h. This gives a contradiction. Thus $(I - T)(L^2(S))$ is closed, as it was claimed.

Since the range of $I - T$ is both open and closed, $I - T$ is onto and, consequently, for every $h \in L^2(S)$ Eq. (2) has a (precisely one) solution.

2. As a simple corollary of our Theorem we prove an existence result for equation (2) when f is decreasing in x. The classical results of this type are due to Hammerstein [2], and Krasnoselskii [3].

Our theorem enables us to prove the existence of a unique solution without the assumption that the kernel K is positive definite and symmetric, which was the typical requirement of the classical theory.

COROLLARY. *Let* $K \in L^2(S \times S)$ *and let* f *be as in the Theorem. Assume, moreover, that for some positive* a, b *and* λ

$$-b \leq (f(t, x) - f(t, y))(x - y)^{-1} \leq -a < 0, \quad x, y \in R, \tag{5}$$

$$\int_S \int_S K(s, t) x(s) x(t) \, \mu(ds) \, \mu(dt) \geq -\lambda \| x \|^2, \quad x \in L^2(S) \tag{6}$$

and $\lambda < a b^{-2}$. *Then for every* $h \in L^2(S)$ *equation* (2) *admits exactly one solution.*

Proof. Fix a, b and f satisfying (5). Consider the set $\mathcal{K}$ of those kernels which satisfy (6) with some $\lambda_K < ab^{-2}$. This set is open in $L^2(S \times S)$. We show that for every $K \in \mathcal{K}$ equation (2) has at most one solution (for an arbitrary $h \in L^2(S)$). Suppose there are two distinct solutions x_1 and x_2 of equation (2) with the same h. Set $y = x_1 - x_2$ and $q(s) = (f(s, x_1(s)) - f(s, x_2(s)))(x_1(s) - x_2(s))^{-1}$ if $x_1(s) - x_2(s) \neq 0$ and $-a$ otherwise. Then

$$y(t) = \int_S K(s, t) q(s) y(s) \mu(ds).$$

Multiplying the above quality by $q(t) y(t)$ and integrating over S we get

$$\int_S y^2(t) q(t) \mu(dt) = \int_S \int_S K(s, t) q(s) y(s) q(t) y(t) \mu(ds) \mu(dt).$$

Assumptions (5) and (6) imply that

$$\int_S y^2(t) q(t) \mu(dt) < - \|y\|^2 a$$

$$\int_S \int_S K(s, t) q(s) y(s) q(t) y(t) \mu(ds) \mu(dt) \geq -\lambda \|yq\|^2 \geq -\lambda \|y\|^2 b^2$$

and consequently $\lambda \geq ab^{-2}$. This contradics the definition of $\mathcal{K}$.

Now an application of our Theorem completes the Proof.

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