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Nonoscillation Criteria for Elliptic Equations of Order 2 m

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Equazioni differenziali a derivate parziali. — *Nonoscillation Criteria for Elliptic Equations of Order 2 m.* Nota di E. S. NOUSSAIR e N. YOSHIDA, presentata (*) del Socio B. SEGRE.

RIASSUNTO. — Vengono stabiliti criteri di non oscillazione per equazioni a derivate parziali ellittiche d'ordine pari.

Nonoscillation criteria for elliptic operators have been developed by many authors. We refer, in particular, to Glazman [2], Headley [3], Headley and Swanson [4], Kreith [5], Kuks [6], Kusano and Yoshida [7], Noussair [10], Piepenbrink [11], Skorobogat'ko [12], Swanson [13] and Yoshida [14] for second order elliptic equations or systems, and to Kusano and Yoshida [8] and Yoshida [15] for fourth order elliptic equations or systems. In this paper we establish nonoscillation criteria for elliptic equations of even order.

Consider the linear elliptic operator L defined by

$$\text{(1)} \qquad \mathrm{L}u = (-)^m \sum_{|\beta|=|\alpha|=m} \mathrm{D}^\alpha (a_{\alpha\beta}(x)\, \mathrm{D}^\beta u) - c(x)\, u\,,$$

where coefficients are defined in an unbounded domain R of n-dimensional Euclidean space E^n. The boundary $\partial\mathrm{R}$ of R is supposed to have a piecewise continuous unit normal vector at each point.

Points of E^n are denoted by $x = (x_1, \cdots, x_n)$. The differential operator D^α is defined as usual by

$$\mathrm{D}^\alpha = \mathrm{D}_1^{\alpha(1)} \cdots \mathrm{D}_n^{\alpha(n)}\,, \quad \alpha = (\alpha(1), \cdots, \alpha(n))\,, \quad |\alpha| = \sum_{i=1}^{n} \alpha(i)\,,$$

where each $\alpha(i)$, $i = 1, \cdots, n$, is a nonnegative integer. The coefficients $a_{\alpha\beta}$ are symmetric, i.e., $a_{\alpha\beta} = a_{\beta\alpha}$, and smooth enough to make all the partial derivatives involved in L exist and be continuous in $\overline{\mathrm{R}}$, the closure of R.

A bounded domain $\mathrm{N} \subset \mathrm{R}$ is said to be a *nodal domain* for L if there exists a nontrivial function $w \in \mathrm{C}^{2m}(\mathrm{N}) \cap \mathrm{C}^m(\overline{\mathrm{N}})$ such that $\mathrm{L}w = 0$ in N, $\mathrm{D}^\alpha w = 0$ on $\partial\mathrm{N}$ for all α with $|\alpha| \leq m - 1$.

The operator L is said to be *oscillatory* in R if it has a nodal domain outside of every sphere centred at the origin.

The operator L is said to be *nonoscillatory* in R if it is not oscillatory in R, i.e., if there exists a number $r > 0$ such that it has no nodal domain in R_r, where

$$\mathrm{R}_r = \mathrm{R} \cap \{x \in \mathrm{E}^n : |x| > r\}.$$

(*) Nella seduta del 12 aprile 1975.

Let the finite set of multi-indices α be ordered in an arbitrary manner, in a sequence $S = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, where $\alpha_i = (\alpha_i(1), \alpha_i(2), \cdots, \alpha_i(n))$. Each $\alpha_i(q)$ $(i = 1, 2, \cdots, k)$ $(q = 1, 2, \cdots, n)$ is a nonnegative integer, $\sum_{q=1}^{n} \alpha_i(q) = m$, and k is the number of the multi-indices α. We can arrange the coefficients $a_{\alpha\beta}$ in the form of a $k \times k$ matrix M_S defined by

$$M_S = (a_{\alpha_i \alpha_j}), \qquad i, j = 1, 2, \cdots, k.$$

It is easy to verify that the sets of eigenvalues of all matrices obtained using any permutation of the sequence S are equal.

Let $\lambda(x)$ denote the smallest eigenvalue of the coefficient matrix M_S; and let σ denote a subset of the set of all multi-indices α with $|\alpha| = m$, and L_σ be the operator defined by

$$L_\sigma u = (-)^m \sum_{\alpha \in \sigma} D^\alpha (\lambda(x) D^\alpha u) - c(x) u. \tag{2}$$

THEOREM 1. *If for any choice of σ the operator L_σ is nonoscillatory in R, then the operator L is nonoscillatory in R.*

Proof. Suppose to the contrary that L_σ is nonoscillatory but L is oscillatory. Then we can choose $r > 0$ such that L_σ has no nodal domain in R_r. Since L is oscillatory, there is a nodal domain $N_r \subset R_r$ and a nontrivial solution u_r of the boundary value problem

$$Lu_r = 0 \quad \text{in } N_r, \quad D^\alpha u_r = 0 \quad \text{on } \partial N_r \text{ for } |\alpha| \leqq m - 1.$$

It is easy to get the following

$$\int_{N_r} u_r L_\sigma u_r \, dx = \int_{N_r} \left[\sum_{\alpha \in \sigma} \lambda(x) (D^\alpha u_r)^2 - c(x) u_r^2\right] dx \leqq \int_{N_r} u_r L u_r \, dx = 0.$$

Hence, the smallest eigenvalue of the problem

$$L_\sigma u = \lambda u \quad \text{in } N_r, \quad D^\alpha u = 0 \quad \text{on } \partial N_r \text{ for } |\alpha| \leqq m - 1$$

is nonpositive by a well known result (see, e.g., Allegretto and Swanson [1]).

Since the smallest eigenvalue increases to infinity as N_r shrinks to the empty set (see Noussair [8]), there exists a domain $N_r \subset R_r$ which is a nodal domain for the operator L_σ. This contradicts our assumption and proves the Theorem.

The following integral inequality is known [2, pp. 83].

LEMMA. *For any real-valued function $u(t) \in C_0^m(0, \infty)$*

$$\int_0^\infty x^{-2m} u^2(x) \, dx \leqq \frac{2^{2m}}{[(2m-1)!!]^2} \int_0^\infty \left(\frac{d^m u}{dx^m}\right)^2 dx,$$

where $(2m-1)!! = (2m-1)(2m-3), \cdots, 1$.

Proof. We can easily obtain the integral identity

$$\int_0^\infty x^{-2m} u^2(x)\,dx = 2\int_0^\infty x^{-2m}\,dx \int_0^x u(t)\frac{du}{dt}\,dt = \frac{2}{2m-1}\int_0^\infty t^{1-2m} u(t)\frac{du}{dt}\,dt.$$

Hence, by means of the Cauchy-Schwarz inequality we have

$$\int_0^\infty x^{-2m} u^2(x)\,dx \leqq \frac{2^2}{(2m-1)^2}\int_0^\infty x^{2m-2}\left(\frac{du}{dx}\right)^2 dx.$$

Repeating the above procedure, we obtain the desired inequality.

For the following nonoscillation theorem we shall assume that the domain R is contained in a cone with vertex angle less than π, and that R contains a half line. Without loss of generality assume that

(i) $$R \subset C_\alpha = \{x \in E^n : x^1 \geq |x| \cos\alpha\}$$

for some α, $0 < \alpha \leq \pi$

(ii) $$\{x = (x_1, 0) \in R^n\} \subset R.$$

NOTATION. Let

$$c^+(x) = \max\{c(x), 0\},$$

$$g(t) = \sup\{c^+(x), x = (t, \bar{x}) \in R\}, \qquad \bar{x} = (x_2, \cdots, x_n).$$

THEOREM 2. *Let* $\lambda(x)$ *be bounded below in* R *by some positive number* λ_0. *Then the operator* L *is nonoscillatory in* R *if*

$$\limsup_{t\to\infty} t^{2m-1}\int_t^\infty g(t)\,dt < \frac{\alpha_m^2}{2m-1}\lambda_0,$$

where $\alpha_m = \dfrac{(2m-1)!!}{2^m}$.

Proof. Choose $\delta > 0$ large enough such that for all $(t, \bar{x}) \in R_\delta$,

$$t^{2m-1}\int_t^\infty g(t)\,dt < \frac{\alpha_m^2}{2m-1}\lambda_0.$$

This is possible by hypothesis and the assumptions made on the domain R. Suppose to the contrary that L is oscillatory in R. Then there exists a nontrivial solution v_δ of (1) with a nodal domain $N_\delta \subset R_\delta$. Thus

$$\begin{aligned} 0 &= \int_{N_\delta} v_\delta L v_\delta \, dx \\ &\geq \lambda_0 \int_{N_\delta} \left[\sum_{|\alpha|=m} (D^\alpha v_\delta)^2 - \frac{c(x)}{\lambda_0} v_\delta^2 \right] dx \\ &\geq \lambda_0 \int_{N_\delta} \left[\left(\frac{\partial^m v_\delta}{\partial x_1^m} \right)^2 - \frac{g(x_1)}{\lambda_0} v_\delta^2 \right] dx . \end{aligned}$$

Let u_δ be the extension of v_δ to all of E^n which is identically zero outside N_r. Then

$$\int_{N_\delta} \left[\left(\frac{\partial^m v_\delta}{\partial x_1^m} \right)^2 - \frac{g(x_1)}{\lambda_0} v_\delta^2 \right] dx = \int_{E^{n-1}} \int_\delta^\infty \left[\left(\frac{\partial^m u_\delta}{\partial x_1^m} \right)^2 - \frac{g(x_1)}{\lambda_0} u_\delta^2 \right] dx_1 \, d\bar{x} .$$

However

$$\int_\delta^\infty g(x_1) u_\delta^2 (x_1, \bar{x}) \, dx_1 = 2 \int_\delta^\infty g(x_1) \, dx_1 \int_\delta^{x_1} u_\delta \frac{\partial u_\delta}{\partial y_1} dy_1 \tag{4}$$

$$= 2 \int_\delta^\infty u_\delta \frac{\partial u_\delta}{\partial y_1} dy_1 \int_{y_1}^\infty g(x_1) \, dx_1 \leq 2 \int_\delta^\infty \left| u_\delta \frac{\partial u_\delta}{\partial y_1} \right| y_1^{1-2m} y_1^{2m-1} dy_1 \int_{y_1}^\infty g(x_1) \, dx_1 .$$

By the choice of δ and the above inequality (4), we get

$$\int_\delta^\infty g(x_1) u_\delta^2 \, dx_1 < \frac{2 \alpha_m^2}{2m-1} \lambda_0 \left\{ \int_\delta^\infty y_1^{-2m} u_\delta^2 \, dy_1 \right\}^{1/2} \left\{ \int_\delta^\infty y_1^{2-2m} \left(\frac{\partial u_\delta}{\partial y_1} \right) dy_1 \right\}^{1/2} .$$

From the last inequality (5) and the Lemma, we have

$$\int_\delta^\infty \frac{g(x_1)}{\lambda_0} u_r^2 \, dx_1 \leq \int_\delta^\infty \left(\frac{\partial^m u_r}{\partial y_1^m} \right) dy_1$$

which contradicts inequality (3). This completes the proof.

By choosing the set σ so that the operator L_σ has a simple form, we can obtain several nonoscillation criteria for the operator L.

Suppose $m = 2p$. Then we can choose L_σ as

$$L_\sigma u = \Delta^p (\lambda(x) \Delta^p u) - c(x) u.$$

DEFINITION. The elliptic operator L_0 definid by

$$L_0 u = \Delta^p (\Lambda(x) \Delta^p u) - C(x) u,$$

where $\Lambda(x) > 0$ in R, is said to belong to $M[L_\sigma; R_r]$ for some $r > 0$, if for every bounded subdomain Ω of R_r the functional

$$V[u; \Omega] \equiv \int_\Omega [(\lambda - \Lambda)(\Delta^p u)^2 + (C - c) u^2] \, dx$$

is nonnegative for all $u \in C^{2p}(\overline{\Omega})$ such that $D^\alpha u = 0$ on $\partial\Omega$ for all $|\alpha| \leqq 2p - 1$.

THEOREM 3 ($m = 2p$). *The operator* L *is nonoscillatory in* R *if, for some* $r > 0$, *there exists an elliptic operator* $L_0 \in M[L_\sigma; R_r]$ *and vector functions* $(\varphi_1, \cdots, \varphi_{2p})$, $(\psi_1, \cdots, \psi_p)$ *of class* $C^2(R_r)$ *such that*

(i) $\varphi_l < 0$ *in* R_r, $1 \leqq l \leqq p - 1$, $\varphi_l \leqq 0$ *in* R_r, $p \leqq l \leqq 2p - 2$,

$\varphi_{2p-1} < 0$ [*resp.* $\leqq 0$] *in* R_r, $\Lambda\varphi_{2p} - C \geqq 0$ [*resp.* > 0] *in* R_r,

(ii) $\Delta(\Lambda\varphi_{p+l}) + \Lambda\varphi_{p+l}\varphi_{p-l} + 2\nabla(\Lambda\varphi_{p+l}) \cdot \nabla\psi_{p-l} \geqq \Lambda\varphi_{p+l+1}/\varphi_{p-l-1}$

in R_r, $0 \leqq l \leqq p - 1$,

(iii) $-\Delta\psi_{p-l} - |\nabla\psi_{p-l}|^2 + \varphi_{p-l} \geqq 0$ *in* R_r, $0 \leqq l \leqq p - 1$,

(iv) *For every bounded subdomain* Ω *of* R_r *the relation*

$$\nabla u - u\nabla\psi_1 \not\equiv 0 \quad \textit{in } \Omega$$

holds for all nontrivial $u \in C^{2p}(\overline{\Omega})$ *such that* $D^\alpha u = 0$ *on* $\partial\Omega$ *for all* $|\alpha| \leqq 2p - 1$.

Proof. Theorem 1 implies that it is sufficient to prove that L_σ is nonoscillatory in R. Suppose to the contrary that L_σ is oscillatory in R. Then there exists a nodal domain $N_r \subset R_r$ and a nontrivial solution u_r of the boundary value problem

$$L_\sigma u_r = 0 \quad \text{in } N_r, \quad D^\alpha u_r = 0 \quad \text{on } \partial N_r \quad \text{for } |\alpha| \leqq 2p - 1.$$

By the hypothesis $L_0 \in M[L_\sigma; R_r]$ and Green's formula we have

$$0 = \int_{N_r} u_r L_\sigma u_r \, dx \geqq \int_{N_r} [\Lambda(\Delta^p u_r)^2 - C u_r^2] \, dx. \tag{6}$$

Applying Green's formula, we get the following identities:

$$0 = \int_{N_r} \sum_{l=0}^{p-1} \sum_{i=1}^{n} D_i [D_i(\Lambda\varphi_{p+l})(\Delta^{p-l-1} u_r)^2 - D_i((\Delta^{p-l-1} u_r)^2)\Lambda\varphi_{p+l}] \, dx \tag{7}$$

$$= \int_{N_r} \sum_{l=0}^{p-1} [(\Delta^{p-l-1} u_r)^2 \Delta(\Lambda\varphi_{p+l}) - 2\Lambda\varphi_{p+l}(\Delta^{p-l-1} u_r)\Delta^{p-l} u_r -$$

$$- 2\Lambda\varphi_{p+l} |\nabla(\Delta^{p-l-1} u_r)|^2] \, dx,$$

$$(8)\qquad 0 = 2\int_{N_r}\sum_{l=0}^{p-1}\sum_{i=1}^{n} D_i\,[\Lambda\varphi_{p+l}\,(\Delta^{p-l-1}\,u_r)^2\,D_i\,\psi_{p-l}]\,dx$$

$$= 2\int_{N_r}\sum_{l=0}^{p-1}[(\Delta^{p-l-1}\,u_r)^2\,\nabla(\Lambda\varphi_{p+l})\cdot\nabla\psi_{p-l} + \Lambda\varphi_{p+l}\,(\Delta^{p-l-1}\,u_r)^2\,\Delta\psi_{p-l} +$$

$$+ 2\,\Lambda\varphi_{p+l}\,(\Delta^{p-l-1}\,u_r)\,\nabla\psi_{p-l}\cdot\nabla(\Delta^{p-l-1}\,u_r)]\,dx\,.$$

From (7) and (8) we obtain the integral indentity

$$(9)\qquad \int_{N_r}[\Lambda(\Delta^p\,u_r)^2 - Cu_r^2]\,dx = \int_{N_r}\Bigg[\Lambda(\Delta^p\,u_r - \varphi_p\,\Delta^{p-1}\,u_r)^2 +$$

$$+\sum_{l=0}^{p-2}\frac{\Lambda\varphi_{p+l+1}}{\varphi_{p-l-1}}(\Delta^{p-l-1}\,u_r - \varphi_{p-l-1}\,\Delta^{p-l-2}\,u_r)^2 + (\Lambda\varphi_{2p} - C)\,u_r^2\Bigg]\,dx +$$

$$+\int_{N_r}\sum_{l=0}^{p-1}\Bigg[\Delta(\Lambda\varphi_{p+l}) + \Lambda\varphi_{p+l}\,\varphi_{p-l} + 2\,\nabla(\Lambda\varphi_{p+l})\cdot\nabla\psi_{p-l} -$$

$$-\frac{\Lambda\varphi_{p+l+1}}{\varphi_{p-l-1}}\Bigg](\Delta^{p-l-1}\,u_r)^2\,dx - 2\int_{N_r}\sum_{l=0}^{p-1}\Lambda\varphi_{p+l}\,[\,|\,\nabla(\Delta^{p-l-1}\,u_r)\,|^2 -$$

$$-2\,(\Delta^{p-l-1}\,u_r)\,\nabla\psi_{p-l}\cdot\nabla(\Delta^{p-l-1}\,u_r) + (-\Delta\psi_{p-l} + \varphi_{p-l})\,(\Delta^{p-l-1}\,u_r)^2]\,dx,$$

where we have set $\varphi_0 = 1$. We note that the integrand of the third integral on the right hand side of (9) can be rewritten as follows:

$$\sum_{l=0}^{p-1}\Lambda\varphi_{p+l}\,[\,|\,\nabla(\Delta^{p-l-1}\,u_r)\,|^2 - 2\,(\Delta^{p-l-1}\,u_r)\,\nabla\psi_{p-l}\cdot\nabla(\Delta^{p-l-1}\,u_r) +$$

$$+ (-\Delta\psi_{p-l} + \varphi_{p-l})\,(\Delta^{p-l-1}\,u_r)^2]$$

$$= \sum_{l=0}^{p-1}\Lambda\varphi_{p+l}\,[\,|\,\nabla(\Delta^{p-l-1}\,u_r) - \Delta^{p-l-1}\,u_r\,\nabla\psi_{p-l}\,|^2 + (-\Delta\psi_{p-l} -$$

$$-\,|\,\nabla\psi_{p-l}\,|^2 + \varphi_{p-l})\,(\Delta^{p-l-1}\,u_r)^2]\,.$$

Hence, the hypotheses imply that the right hand side of (9) is positive. This contradicts (6) and completes the proof.

THEOREM 4 ($m = 2p$). *The operator* L *is nonoscillatory in* R *if, for some* $r > 0$, *there exists an elliptic operator* $L_0 \in M\,[L_\sigma\,;\,R_r]$ *and a function* $w \in C^{4p}(R_r)$ *such that*

(i) $(-1)^l\,\Delta^l w > 0$ *in* $\bar{R}_r$, $0 \leq l \leq p-1$,

(ii) $(-1)^{p+l}\,\Delta^l(\Lambda\Delta^p w) \geq 0$ *in* R_r, $0 \leq l \leq p-2$,

(iii) $\Delta^{p-1}(\Lambda\Delta^p w) < 0$ [*resp.* ≤ 0] *in* R_r,

(iv) $L_0 w \geq 0$ [*resp.* > 0] *in* R_r, *Proof.* Define the vector function $(\varphi_1, \cdots, \varphi_{2p})$ and $(\psi_1, \cdots, \psi_p)$ by

$$\varphi_l = \Delta^l w / \Delta^{l-1} w \quad , \quad 1 \leqq l \leqq p \quad , \quad \varphi_{p+l} = \Delta^l (\Lambda \Delta^p w) / (\Lambda \Delta^{p-l-1} w) \, , \qquad 0 \leqq l \leqq p-1 \, ,$$

$$\varphi_{2p} = \Delta^p (\Lambda \Delta^p w) / (\Lambda w) \quad , \quad \psi_l = \log ((-1)^{l-1} \Delta^{l-1} w) \, , \qquad 1 \leqq l \leqq p \, .$$

It is easy to see that the following identities hold:

$$\Delta (\Lambda \varphi_{p+l}) + \Lambda \varphi_{p+l} \varphi_{p-l} + 2 \nabla (\Lambda \varphi_{p+l}) \cdot \nabla \psi_{p-l} = \Lambda \varphi_{p+l+1} / \varphi_{p-l-1} \, , \qquad 0 \leqq l \leqq p-1 \, ,$$

$$-\Delta \psi_{p-l} - | \nabla \psi_{p-l} |^2 + \varphi_{p-l} = 0 \, , \qquad 0 \leqq l \leqq p-1 \, ,$$

$$\Lambda \varphi_{2p} - C = L_0 w / w \quad , \quad \nabla u - u \nabla \psi_1 = w \nabla (u/w) \, .$$

Hence, the conclusion follows from Theorem 3.

THEOREM 5 $(m = 2p)$. *Let* $\lambda(x)$ *be bounded below in* R *by some positive number* λ_0. *Then the operator* L *is nonoscillatory in* R *if*

$$\limsup_{r \to \infty} \omega(r) < 0 \qquad \textit{for} \quad 2p < n \leqq 4p \, ,$$

$$\limsup_{r \to \infty} r^{4p} \omega(r) < \prod_{i=1}^{2p} (n/2 + 2p - 2i)^2 \lambda_0 \qquad \textit{for} \quad n > 4p \, ,$$

where $\omega(r) = \max_{x \in S_r} c(x)$ *and* $S_r = \bar{R} \cap \{x \in E^n : |x| = r\}$.

Proof. Taking

$$L_0 u = \lambda_0 \Delta^{2p} u - \omega(|x|) u \, ,$$

we see that $L_0 \in M[L_\sigma ; R_r]$ for some $r > 0$. The function $w = |x|^s$ satisfies

$$\Delta^l w = \prod_{i=0}^{l-1} (s - 2i) \times \prod_{j=1}^{l} (s + n - 2j) |x|^{s-2l}, \qquad 1 \leqq l \leqq 2p \, ,$$

$$L_0 w = (\lambda_0 h(s) - |x|^{4p} \omega(|x|)) |x|^{s-4p},$$

where $h(s) = \prod_{i=0}^{2p-1} (s - 2i) \times \prod_{j=1}^{2p} (s + n - 2j)$. Letting $s = 2p - n \; (2p < n)$, it si easy to see that

$$(-1)^l \Delta^l w > 0 \qquad \text{in} \quad \bar{R}_r \, (0 \leqq l \leqq p-1) \, ,$$

$$(-1)^{p+l} \Delta^{p+l} w = 0 \qquad \text{in} \quad R_r \, (0 \leqq l \leqq p-1) \, ,$$

$$L_0 x = -\omega(|x|) |x|^{2p-n}.$$

Consider the case $n > 4p$. Putting $s = 2p - n/2$, we see that

$$(-1)^l \Delta^l w > 0 \quad \text{in} \quad \bar{R}_r \, (0 \leqq l \leqq 2p - 1),$$

$$h(2p - n/2) = \prod_{i=1}^{2p} (n/2 + 2p - 2i)^2.$$

Hence, the conclusion follows from Theorem 4.

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