Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Application of the contracting mapping principle to a system of Duffing type

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.1-2, p. 51–56.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_1-2_51_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Equazioni differenziali ordinarie. — *Application of the contracting mapping principle to a system of Duffing type.* Nota (*) di ROLF REISSIG, presentata dal Socio G. SANSONE.

RIASSUNTO. — S. Ahmad e A. C. Lazer hanno provato l'esistenza di una e una sola soluzione periodica di un sistema conservativo periodico perturbato del tipo di G. Duffing nell'ipotesi che il potenziale delle forze di richiamo soddisfi una condizione abbastanza generale escludente il caso di risonanza.

In questa Nota l'A. prova che una condizione simile consente l'applicazione del teorema del punto fisso di Banach a problemi più generali contenenti un termine di smorzamento.

La dimostrazione si fonda su un metodo proposto da J. J. Mawhin esteso dall'A. in una precedente sua Nota.

This paper is devoted to the problem of periodic solutions of the vector differential equation

$$x'' + cx' + \operatorname{grad} G(x) = e(t) \equiv e(t + 2\pi), \; x \in \mathbf{R}^n: \tag{1}$$

$$x(t + 2\pi) \equiv x(t), \text{ i.e. } x^{(i)}(0) = x^{(i)}(2\pi) \qquad (i = 0, 1).$$

Here c is an arbitrary real constant, $e(t) \in C^0(\mathbf{R})$ and $G(x) \in C^2(\mathbf{R}^n)$ with the Hessian matrix $H(x) = H^*(x)$.

Our aim is to derive a condition for this matrix under which there is exactly one periodic solution to be determined by Picard's iteration. This condition is suggested by a paper of Lazer [4] treating the uniqueness problem; the corresponding existence problem is solved in a subsequent paper of Ahmad [1].

Both authors consider system (1) in the special case when it is conservative ($c = 0$). Their result is as follows.

Let A and B be real symmetric $n \times n$ matrices with eigenvalues $\lambda_1(A) \leqq \cdots \leqq \lambda_n(A)$ and $\lambda_1(B) \leqq \cdots \leqq \lambda_n(B)$, respectively. Assume $A \leqq B$ a consequence of which is $\lambda_i(A) \leqq \lambda_i(B)$ for all indices i. Moreover suppose that there are non-negative integers m_i such that

$$m_i^2 < \lambda_i(A) \leqq \lambda_i(B) < (m_i + 1)^2, \qquad 1 \leqq i \leqq n. \tag{2}$$

Then in case

$$A \leqq H(x) \leqq B \qquad \forall x \in \mathbf{R}^n \tag{3}$$

there exists at most one periodic solution (see Lazer [4]), and the existence of such a solution is ensured, too (see Ahmad [1]). Evidently, we can assume that $\lambda_i = \lambda_j$ for $m_i = m_j$.

(*) Pervenuta all'Accademia il 7 luglio 1975.

Our study is based on the restriction that A and B are commutative, AB = BA; then there is an orthogonal matrix $T\,(T^{-1} = T^*)$ such that

$$TAT^* = \operatorname{diag}(a_1, \cdots, a_n),$$

$$TBT^* = \operatorname{diag}(b_1, \cdots, b_n)$$

where $a_i = \lambda_i(A)$ and $b_i = \lambda_i(B)$. Substituting $y = Tx$ in equation (1) the Hessian matrix H (x) is transformed like its bounds A and B, and the estimate (3) remains valid. Thus, without loss of generality, we can suppose that A and B are of diagonal form. Then condition (2) can be replaced by

$$M < A < B \leqq M' \tag{4}$$

where the notations $M = \operatorname{diag}(\mu_1, \cdots, \mu_n)$, $M' = \operatorname{diag}(\mu_1', \cdots, \mu_n')$ and $\mu_i = m_i^2$, $\mu_i' = (m_i + 1)^2$ are used. Choosing an adequate number $\vartheta \in (0, 1)$ conditions (3)–(4) can be weakened to

$$-\frac{1-\vartheta}{2}(M' - M) \leqq N - H(x) \leqq \frac{1-\vartheta}{2}(M' - M) \qquad \forall x \in \mathbf{R}^n \tag{5}$$

where $N = \frac{M' + M}{2} = \operatorname{diag}(\upsilon_1, \cdots, \upsilon_n)$. However, our argument makes no use of condition (5) but of the similar one

$$(N - H(x))^2 \leqq \frac{(1-\vartheta)^2}{4}(M' - M)^2 \qquad \forall x \in \mathbf{R}^n . \tag{6}$$

Moreover, the class of admissible pairs of matrices (M, M') is more comprehensive than above:

$$\mu_1 \leqq \cdots \leqq \mu_n \qquad \text{with} \quad \mu_i < 0\ (\mu_i' = 0) \qquad \text{or} \quad \mu_i = m_i^2\ (\mu_i' = (m_i + 1)^2).$$

By means of Mawhin's Hilbert space method (see [6]) extended in a previous note (see [7]) we prove the following theorem:

The differential equation (1) *satisfying condition* (6) *possesses a uniquely determined periodic solution which can be constructed by means of Picard's iteration.*

Remark. $H = H^*$ being constant conditions (5) and (6) are fulfilled if

$$-\frac{1}{2}(M' - M) < N - H < \frac{1}{2}(M' - M) \tag{5'}$$

or

$$(N - H)^2 < \frac{1}{4}(M' - M)^2, \tag{6'}$$

respectively. Each of these estimates is independent of the other one (see Bellman [2] as well as Löwner [5] and Dobsch [3]).

Let, for example, $n = 2$, $\mu_1 = 0$, $\mu_2 = 1$ and $H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Then (5′) is valid iff

$$0 < a < 1 \,,\; b^2 < \min \{a (c - 1) \,,\, (1 - a)(4 - c)\} \,,$$

and (6′) is valid iff

$$\left(a - \frac{1}{2}\right)^2 + b^2 < \frac{1}{4} \,,\; \left[\left(a - \frac{1}{2}\right)^2 + b^2 - \frac{1}{4}\right] \left[\left(c - \frac{5}{2}\right)^2 + b^2 - \frac{9}{4}\right] >$$
$$> b^2 (a + c - 3)^2 \,.$$

Choosing $a = \frac{1}{2}$, $c = \frac{5}{2}$, $b = \frac{1}{2}$ we satisfy (5′) but not (6′);

choosing $a = \frac{1}{2}$, $c = \frac{5}{2} + \sqrt{2}$, $b = \frac{1}{4}$ we satisfy (6′) but not (5′).

Conditions (5) and (6) are in accordance when $H(x)$ is of diagonal form, $H = \operatorname{diag}(h_1, \cdots, h_n)$, or when $\mu_1 = \mu_n = \mu$ $(\mu_1' = \mu_n' = \mu')$: In the first case the conditions mean that

$$|\, \upsilon_i - h_i \,| \leqq \frac{1 - \vartheta}{2} (\mu_i' - \mu_i) \qquad \text{for all indices } i.$$

In the second case an orthogonal matrix $S(x)$ is chosen such that $SHS^* = \operatorname{diag}(h_1, \cdots, h_n)$; $N - H$ and $M' - M$ are replaced by $S(N - H) S^* = N - SHS^*$ and $S(M' - M) S^* = M' - M$, respectively. Then (5) as well as (6) is equivalent to

$$|\, \upsilon - h_i \,| \leqq \frac{1 - \vartheta}{2} (\mu' - \mu) \qquad \text{for} \quad 1 \leqq i \leqq n \,,\; \upsilon = \frac{1}{2} (\mu' + \mu) \,.$$

In the proof of the theorem we denote $x(t) = \operatorname{col}(x_1(t), \cdots, x_n(t))$; to begin with let us consider the auxiliary problem for the i—th component:

$$(7) \qquad \begin{cases} x_i'' + c x_i' + \upsilon_i x_i = e_i(t) \in C^0 [0 \,,\, 2\pi] \\ x_i(0) = x_i(2\pi) \,,\, x_i'(0) = x_i'(2\pi) \,. \end{cases}$$

As is well-known it admits one and only one solution $x_i(t) \in C^2 [0 \,,\, 2\pi]$ which can be represented by means of a Green function.

The same is true with its first derivative. Now, the periodic boundary value problem is generalized in as much as the right hand member of the differential equation is replaced by an arbitrary function $y_i(t)$ of the real Hilbert space $H = L^2 [0 \,,\, 2\pi]$ where the scalar product $(u_i, v_i) = (2\pi)^{-1} \int_0^{2\pi} u_i(t) \, v_i(t) \, dt$ is used. The solution $x_i(t)$ is to be contained in the subspace $D \subset H$ the elements of which possess first and second order Lebesgue derivatives in H; moreover they satisfy the periodic boundary conditions. Accordingly let us replace (7) by the operator equation

$$A_i x_i = y_i \in H \ (x_i \in D) . \tag{8}$$

The linear operator $A_i : D \to H$ is one-to-one, its inverse is a bounded and compact operator with the norm

$$\| A_i^{-1} \| = \sup ((m^2 - \upsilon_i)^2 + c^2 m^2)^{-1/2}$$

where $m \geqq 0$ an arbitrary integer, see [7]. An immediate consequence is the estimate

$$\| A_i^{-1} \| \leqq \alpha_i = \frac{2}{\mu_i' - \mu_i} .$$

Then we consider the Hilbert space $\hat{H} = \{x(t) : x_i(t) \in H , 1 \leqq i \leqq n\}$ supplied with the scalar product $(u , v)_{\hat{H}} = \sum_{i=1}^{n} \alpha_i^{-2} (u_i , v_i)_H$. Let the subspace $\hat{D}$ consist of all $x(t) \in \hat{H}$ whose components are in D. We introduce the linear mapping $\hat{D} \to \hat{H}$:

$$x \mapsto L x = \operatorname{col} (A_1 x_1 , \cdots , A_n x_n);$$

it has a bounded inverse defined on $\hat{H}$,

$$y \mapsto L^{-1} y = \operatorname{col} (A_1^{-1} y_1 , \cdots , A_n^{-1} y_n) .$$

By virtue of assumption (6) the continuous mapping $\mathbf{R}^n \to \mathbf{R}^n$:

$$x \mapsto N x - \operatorname{grad} G(x)$$

is linearly bounded; for this reason $Bx \equiv Nx(t) - \operatorname{grad} G(x(t))$ defines a nonlinear mapping of the Hilbert space $\hat{H}$ into itself. Consequently the periodic boundary value problem belonging to differential equation (1) can be generalized as an operator equation in $\hat{H}$:

$$Lx - Bx = y \in \hat{H} \ (\text{arbitrary}) , x \in \hat{D} \tag{9}$$

or, equivalently,

$$x = L^{-1} Bx + L^{-1} y . \tag{10}$$

Let us show that the composite mapping $L^{-1} B$ is contractive. We note that it possesses a Gâteaux derivative at each point $x \in \hat{H}$, namely the linear operator $L^{-1} B'(x)$ where for all $w(t) \in \hat{H}$

$$(B'(x) w)(t) = N w(t) - H(x(t)) w(t) =$$

$$= \operatorname{col} \left(\upsilon_1 w_1(t) - \sum_{k=1}^{n} h_{1k}(x(t)) w_k(t) , \cdots , \upsilon_n w_n(t) - \sum_{k=1}^{n} h_{nk}(x(t)) w_k(t) \right)$$

$$\text{a.e. in } [0 , 2\pi] , H(x) = (h_{ik}(x)) .$$

u and v being two arbitrary points of $\hat{\mathrm{H}}$, and $\tilde{x}$ being a suitable point on the joining straight line the mean value theorem (see Vainberg [8]) yields:

$$\| \mathrm{L}^{-1} \mathrm{B} u - \mathrm{L}^{-1} \mathrm{B} v \| \leqq \| \mathrm{L}^{-1} \mathrm{B}' (\tilde{x}) \| \, \| u - v \| .$$

Using the abbreviations $\mathrm{N} - \mathrm{H} (\tilde{x} (t)) = (\eta_{ik} (t))$ and $(\mathrm{N} - \mathrm{H}(\tilde{x}(t)))^2 = (\eta_{ik}^{(2)} (t))$ we calculate:

$$\begin{aligned} \| \mathrm{L}^{-1} \mathrm{B}' (\tilde{x}) w \|_{\hat{\mathrm{H}}}^2 &= \sum_{i=1}^{n} \alpha_i^{-2} \left\| \mathrm{A}_i^{-1} \left\{ \sum_{k=1}^{n} \eta_{ik} (t) w_k (t) \right\} \right\|_{\mathrm{H}}^2 \\ &\leqq \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} \eta_{ik} (t) w_k (t) \right\|_{\mathrm{H}}^2 \\ &= (2 \pi)^{-1} \sum_{i=1}^{n} \int_0^{2\pi} \left(\sum_{k,l=1}^{n} \eta_{ik} (t) \eta_{il} (t) w_k (t) w_l (t) \right) \mathrm{d}t \\ &= (2 \pi)^{-1} \int_0^{2\pi} \left(\sum_{k,l=1}^{n} \eta_{kl}^{(2)} (t) w_k (t) w_l (t) \right) \mathrm{d}t \\ &\leqq (2 \pi)^{-1} \int_0^{2\pi} \left(\frac{(1 - \vartheta)^2}{4} \sum_{j=1}^{n} (\mu_j' - \mu_j)^2 w_j^2 (t) \right) \mathrm{d}t \\ &= (1 - \vartheta)^2 \sum_{j=1}^{n} \alpha_j^{-2} \| w_j \|_{\mathrm{H}}^2 = (1 - \vartheta)^2 \| w \|_{\hat{\mathrm{H}}}^2 \end{aligned}$$

by virtue of (6). Hence we obtain

$$\| \mathrm{L}^{-1} \mathrm{B}' (\tilde{x}) w \|_{\hat{\mathrm{H}}} \leqq (1 - \vartheta) \| w \|_{\hat{\mathrm{H}}} \quad \forall w \in \hat{\mathrm{H}} ,$$

i.e.

$$\| \mathrm{L}^{-1} \mathrm{B}' (\tilde{x}) \|_{\hat{\mathrm{H}}} \leqq 1 - \vartheta .$$

It follows that Banach's fixed point theorem is applicable to the mapping $\hat{\mathrm{H}} \to \hat{\mathrm{H}}$:

$$x \mapsto \mathrm{L}^{-1} \mathrm{B} x + \mathrm{L}^{-1} y ;$$

the unique point x^* is the limit of every sequence of successive approximations $\{u_r\}$, $u_{r+1} = \mathrm{L}^{-1} \mathrm{B} u_r + \mathrm{L}^{-1} y$.

Remark. Since the inverse operator $\mathrm{A}_i^{-1} : \mathrm{H} \to \mathrm{D}$, $y_i \mapsto x_i$, can be defined by means of an integral whose kernel is the Green function of the periodic boundary value problem (7) there is an estimate of the kind

$$| x_i (t) | = | (\mathrm{A}_i^{-1} y_i) (t) | \leqq \rho_i \| y_i \|_{\mathrm{H}} .$$

Using the fact that the Hessian H (x) is uniformly bounded we can derive that

$$| x_i^* (t) - u_{i,r+1} (t) | = | A_i^{-1} \{(B (x^*) - B (u_r))_i\} |$$
$$\leqq \rho_i^* \| x^* - u_r \|_{\hat{H}} .$$

Thus, the sequence of approximations $\{u_r\}$ converges uniformly for all $t \in [0 , 2 \pi]$. The same is true with the first derivatives.

Choosing $y (t) = e (t) \in C^0 [0 , 2 \pi]$ in operator equation (9) we obtain the classical solution of the periodic boundary value problem (1).

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