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A Note on Second Order Nonlinear Oscillations

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Equazioni differenziali ordinarie. — *A Note on Second Order Nonlinear Oscillations.* Nota di E. S. NOUSSAIR, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore estendendo alcuni risultati relativi alle equazioni $u''+f(t,u)=0$ trova condizioni sufficienti atte ad assicurare il carattere oscillatorio degli integrali dell'equazione $u''+f(t,u,u')=0$.

1. INTRODUCTION

In this note we shall be concerned with the oscillatory properties of the nonlinear differential equation

$$u''+f(t,u,u')=0 \tag{1}$$

where $f(t,u,u')$ is continuous with respect to the arguments. We tacitly assume that all solutions of (1) may be continued throughout $[0,\infty)$. A non-trivial solution of (1) is called oscillatory if it has arbitrary large zeros.

Several Authors have considered the problem of establishing sufficient criteria to guarantee the oscillation or nonoscillation of all solutions of (1) in the special case $f(t,u,u')=f(t,u)$. We mention in particular the by Atkinson [1], Waltman [10], Bobisud [2], Onose [9], Legatos and Karsatos [5], Wong [11], Macki and Wong [6] and the references therein. Recently R. P. Jahagirdar and B. S. Lalli [3] considered the special case $f(t,u,u')=a(t)f(t)g(u')$, where $a(t)$ is continuous on $[0,\infty)$, $uf(u)>0$ for $u\neq 0$ and $f(0)=0$, and $g(u')>0$ for $|u'|<\infty$. Theorem 1 in [3] states that $u''+a(t)f(u)g(u')=0$ is oscillatory if $f'(u)\geq 0$ and $\int^{\infty} ta(t)\,dt=\infty$. This result is not true as well as some additional results in [3]. The flaw in their proof of Theorem 1, for example, occurs on pp. 377, lines 3–8: They arrive at wrong conclusions regarding the sign of $u'(t)$ using inequality (4). The following example shows that, without further assumptions, the results in [3] can't be true:

Example: The differential equation

$$u''+\frac{1}{4t^2}u=0$$

is nonoscillatory since $u(t)=t^{1/2}$ is a solution. In this example $g(u')=1$, $f(u)=u$ and $a(t)=\frac{1}{4t^2}$. Hence all the hypotheses of Theorem 1 in [3] are satisfied.

(*) Nella seduta dell'11 giugno 1975.

The purpose of this note is to obtain some oscillation criteria for (1) which extend some of the known results for the special case $f(t, u, u') = f(t, u)$.

2. Main Results

Consider the differential equation

$$u'' + f(t, u) g(u') = 0 \tag{2}$$

where the following assumptions are made:

(3) A) $g(u') \geq k > 0$ for $|u'| < \infty$.

B) $a(t) \varphi(u) \leq f(t, u) \leq b(t) \psi(u)$,

where $a(t), b(t)$ are continuous and nonnegative on $[0, \infty)$, $u\varphi(u) > 0$ and $u\psi(u) > 0$ for $u \neq 0$, and $\varphi'(u) \geq 0$, $\psi'(u) \geq 0$ for $|u| < \infty$.

Associated with equation (2) we consider the differential inequalities

$$u'' + ka(t) \varphi(u) \leq 0, \tag{4}$$

$$v'' + kb(t) \psi(v) \geq 0. \tag{5}$$

Lemma 1. *There does not exist any positive number* t_0 *such that inequality* (4) *has a solution* u *which is positive on* $[t_0, \infty)$ *if*

(i) $$\int^{\infty} ta(t)\, \mathrm{d}t = \infty,$$

(ii) $$\int_1^{\infty} \frac{\mathrm{d}u}{\varphi(u)} < \infty.$$

Proof. Assume to the contrary that $u(t) > 0$ on $[t_0, \infty)$. Since $u''(t) \leq 0$ and $u(t) > 0$ for $t \geq t_0$, a standard argument implies that $u'(t) > 0$ for sufficiently large t, say $t \geq t_1$. Define $\mathrm{V}(t) = \frac{t\, u'(t)}{\varphi(u(t))}$ for $t \geq t_1$. A simple calculation shows that

$$\mathrm{V}'(t) \leq -kt\, a(t) + \frac{u'(t)}{\varphi(u(t))}\, .$$

Integrating, we obtain

$$\mathrm{V}(t) - \mathrm{V}(t_1) \leq -k \int_{t_1}^{t} sa(s)\, \mathrm{d}s + \int_{u(t_1)}^{u(t)} \frac{\mathrm{d}u}{\varphi(u)}\, . \tag{6}$$

If we take the limit as $t \to \infty$ in (6) and use hypotheses (i) and (ii) we arrive at the contradiction that $u'(t) < 0$ for large t. This completes the proof of Lemma 1.

The proof of the following lemma is similar to the proof of Lemma 1.

LEMMA 2. *There does not exist any positive number t_0 such that inequality* (5) *has a solution $v(t)$ which is negative on $[t_0, \infty)$ if*

(i) $$\int^{\infty} t b(t)\, dt = \infty$$

(ii) $$\int_{-1}^{-\infty} \frac{dv}{\psi(v)} < \infty .$$

Lemmas 1 and 2 exclude the linear case $\varphi(u) = \psi(u) = u$ because of hypothesis (ii). By weakening hypothesis (i) slightly, condition (ii) becomes redundant.

Suppose $\varphi(u)$ and $\psi(v)$ satisfy the following conditions:

(7) For each $\delta > 0$ there exist positive constants $k_i = k_i(\delta)$, $\gamma_i = \gamma_i(\delta) \geq 1$ the quotient of odd integers, $i = 1, 2$, such that

A) $\dfrac{u^{\gamma_1}}{\varphi(u)} \leq k_1$ for $u \geq \delta$

B) $\dfrac{v^{\gamma_2}}{\psi(v)} \leq k_2$ for $v \leq -\delta$.

Condition (8) is satisfied, for example, if $\varphi(u)$, $\psi(u)$ are linear or superlinear functions.

LEMMA 3. *Assume conditions* (3) *and* (7) *are satisfied. Then there exists no positive number t_0 such that inequality* (4) (*inequality* (5), *respectively*) *has a solution $u(t)$ which is positive* (*negative, respectively*) *on $[t_0, \infty)$ if*
$\int^{\infty} t^{\lambda} a(t)\, dt = \infty \left(\int^{\infty} t^{\lambda} b(t)\, dt = \infty \text{, respectively} \right)$, *for some $\lambda < 1$.*

Proof. Suppose to the contrary that $u(t)$ is a solution of (4) which is positive on $[t_0, \infty)$. As in the proof of Lemma 1 we can show that $u'(t) > 0$ for $t \geq t_1$. A simple calculation shows that

(8) $$\frac{d}{dt}\left(\frac{u(t)}{u'(t)}\right) \geq \frac{k a(t)\, \varphi(u)\, (u)^2}{(u')^2} + 1 .$$

Using (8) we can choose t_2 large enough such that $\dfrac{u'(t)}{u(t)} \leq \dfrac{2}{t}$ for $t \geq t_2$.

Define

$$V(t) = \frac{t^{\gamma}\, u'(t)}{\varphi(u(t))} .$$

A simple calculation shows that

$$\text{(9)} \qquad V'(t) \leq -t^{\lambda} a(t) + \lambda t^{\lambda-1} \frac{u'(t)}{\varphi(u(t))} .$$

Choosing $\delta = u(t_2)$ and using condition (7), we have

$$\frac{u'(t)}{\varphi(u(t))} \leq \frac{u'(t)(u(t))^{\lambda_1}}{(u(t))^{\gamma_1}\varphi(u(t))} \leq \frac{2k_1}{t(u(t))^{\gamma_1-1}} .$$

Using the above inequality and (9) we obtain

$$V'(t) \leq -t^{\lambda} a(t) + \frac{2\lambda k_1 t^{\gamma-2}}{(u(t_2))^{\gamma_1-1}} .$$

Integrating from t_2 to t and taking the limit as $t \to \infty$ we arrive at the contradiction $V'(t) < 0$ as in Lemma 1. The proof of the other part of Lemma 3 is similar and will be omitted.

THEOREM 4. *Assume that condition* (3) *is satisfied. Then equation* (2) *is oscillatory if*

(i) $\int^{\infty} ta(t)\,dt = \int^{\infty} tb(t)\,dt = \infty$

(ii) $\int_{1}^{\infty} \frac{du}{\varphi(u)} < \infty , \int_{-1}^{-\infty} \frac{du}{\psi(u)} < \infty .$

Proof. Assume to the contrary that $u(t)$ is a nonoscillatory solution of (2). Then, for large t, either $u > 0$ and satisfies inequality (4) or $u < 0$ and satisfies inequality (5). Now applying Lemmas 1 and 2 we obtain the desired contradiction.

Remark. In Theorem 4 if $g(u') \equiv 1$ we obtain a result of Waltmann [10].

COROLLARY 5. *In the differential equation*

$$\text{(10)} \qquad u'' + g(u') \sum_{k=1}^{n} a_k(t) f_k(u) = 0$$

suppose that $\varphi(x) \leq f_k(x) \leq \psi(x)$, $k = 1, 2, \cdots, n$, *where* $x\varphi(x) > 0$, $x\psi(x) > 0$ *for* $x \neq 0$ *and* $\varphi'(x) \geq 0$, $\psi'(x) \geq 0$ *for* $|x| < \infty$ *and each* $a_k(t)$ *is nonnegative and* $g(u') \geq k > 0$ *for* $|u'| < \infty$. *Then all solutions of* (10) *are oscillatory if*

(i) $\int^{\infty} t \sum_{k=1}^{n} a_k(t)\,dt = \infty ,$

(ii) $\int_{11}^{\infty} \frac{du}{\varphi(u)} < \infty , \int_{-1}^{-\infty} \frac{dv}{\psi(v)} < \infty .$

Proof. Notice that

$$\varphi(u)\sum_{k=1}^{n} a_k(t) \leq \sum_{k=1}^{n} a_k(t)\, b_k(t) \leq \psi(u)\sum_{k=1}^{n} a_k(t).$$

The conclusion follows from Theorem 3.

Remark. In (10) if we take $g(u') \equiv 1$ and $f_k(u) = u^{2k+1}$ then we obtain a result of Jones [4].

The following theorem covers both the linear and nonlinear cases:

THEOREM 6. *Assume that conditions* (3) *and* (7) *are satisfied. Then equation* (2) *is oscillatory if* $\int^{\infty} t^{\lambda} a(t)\, dt = \infty = \int^{\infty} t^{\lambda} b(t)\, dt$ *for some* $\lambda < 1$.

Proof. If $u(t)$ is a nonoscillatory solution of (2), then, for large t, u is positive and satisfies (4) or u is negative and satisfies (5). The desired conclusions follows from Lemma 3.

Remark. If we restrict Theorem 6 to the linear case $(f(t, u) = a(t)\, u,$ $a(t) \geq 0, g(u') \equiv 1)$ then we obtain a result of Moore [7].

Remark. Condition (3) A can't be weakened to $g(u') > 0$ for $u' \neq 0$ as the following simple example shows:

Example: $u'' + u^3 (u')^2 = 0$ has the nonoscillatory solution $u(t) \equiv 1$. In this example all the hypotheses of Theorem 4 and 6 are satisfied except condition (3) A.

Remark. In Theorems 4 and 6 we can replace condition (3) A by $g(u') > 0$ and $g'(u') \leq 0$ for $|u'| < \infty$ provided we weaken hypothesis (i) in Theorem 4 and the corresponding hypothesis in Theorem 6 to read

$$\int^{\infty} a(t)\, dt = \int^{\infty} b(t)\, dt = \infty.$$

The proof is a slight modification of the proofs used above. In this case we use the function

$$V_1(t) = \frac{u'(t)}{\varphi(u(t))\, g(u')}$$

instead of the functions V in the proofs of Lemmas 1, 2 and 3.

Remark. The other special cases for equation (1) considered in [3] can be handled easily using Lemmas 1, 2 and 3.

Some of the arguments in this note are similar to those used in a recent joint paper by the Author and C. A. Swanson [8].

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