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Minimal displacement of points under weakly inward pseudo-lipschitzian mappings

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Analisi funzionale. — *Minimal displacement of points under weakly inward pseudo-lipschitzian mappings.* Nota (*) di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia C un sottinsieme limitato, chiuso e convesso di uno spazio di Banach $(E, |\,|)$ e sia T una trasformazione continua, debolmente interna pseudo-lipschitziana. In questa Nota si considera $\inf\{|x - Tx| : x \in C\}$. T appartiene a questa classe di rappresentazione se e soltanto se $T - I$ (I indica l'identità) è un generatore fortemente continuo di semigruppo non lineare di C.

INTRODUCTION

Let E be a real Banach space with norm $|\,|$, and let B(E) denote the family of all non-empty bounded closed convex subsets of E. For each C in B(E) the Chebyshev radius of C is $R(C) = \inf\{\sup\{|x - y| : y \in C\} : x \in C\}$. If $k \geq 0$ we denote by $L(C, k)$ the family of those mappings $T : C \to C$ which satisfy a Lipschitz condition with constant k (that is, $|Tx - Ty| \leq k|x - y|$ for all x and y in C). In a recent paper [6], Goebel studied the quantity $\inf\{|x - Tx| : x \in C\}$ for T in $L(C, k)$. He defined, for each E and k, a function $g(E, k) : [0, \infty) \to [0, 1)$ by $g(E, k) = \sup\{\inf\{|x - Tx| : x \in C\}/R(C) : C \in B(E), T \in L(C, k)\}$, and determined some of its properties. (We have, of course, $\inf\{|x - Tx| : x \in C\} \leq g(E, k) R(C)$ for arbitrary C in B(E) and T in $L(C, k)$).

In this note we wish to consider $\inf\{|x - Tx| : x \in C\}$ for continuous weakly inward pseudo-lipschitzian $T : C \to E$ (see the definitions in Section 1 below). This wider class of mapping is important because a mapping T belongs to it if and only if $T - I$ (I stands for the identity) is a continuous (strong) generator of a nonlinear semigroup on C [11, p. 411].

1. PRELIMINARIES

Let k be a real number, and let C be a subset of E. A mapping $T : C \to E$ will be called pseudo-lipschitzian with constant k il for all positive r and x and y in C, $|x - y - r(Tx - Ty)| \geq (1 - rk)|x - y|$.

It is clear that a lipschitzian mapping (with constant k) is pseudo-lipschitzian (with constant k). (The converse is false). A psaudo-contraction [2, p. 876] is a pseudo-lipschitzian mapping with $k = 1$.

Let E^* be the dual space of E. The conjugate norm on E^* will also be denoted by $|\,|$. The duality mapping J from E into the family of non-empty

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weak-star compact convex subsets of E^* is defined by $J(x) = \{x^* \in E^* : (x, x^*) = |x|^2 = |x^*|^2\}$. The following lemma is essentially known [9, p. 509 and 10, p. 416].

LEMMA 1. *If z and w are in* E *and k is real, then the following are equivalent*:

(1) $$|z - rw| \geq (1 - rk)|z| \quad \text{for all positive } r.$$

(2) $$\lim_{r \to 0+} (|z - rw| - |z|)/(-r) \leq k|z|.$$

(3) $$(w, j) \leq k|z|^2 \quad \textit{for some } j \in J(z).$$

Proof. (1) ⇒ (2) is clear because the limit in (2) always exists. In order to prove that (2) ⇒ (3), we may assume that $|z - rw| \neq 0$. Let $j_r \in J(z - rw)$ and $g_r = j_r/|j_r|$. Let a subnet $\{g_t\}$ of $\{g_r\}$ weak-star converge to a certain g in the unit ball of E^*. Since $(w, g_t) \leq (|z - tw| - |z|)/(-t)$, we have $(w, j) \leq k|z|^2$ for $j = |z| g \in J(z)$. Finally, (3) ⇒ (1) because $(1 - rk)|z|^2 = (z, j) - rk|z|^2 \leq (z, j) - (rw, j) = (z - rw, j) \leq |z - rw||z|$.

This lemma immediately implies the following proposition.

PROPOSITION 1. *Let* C *be a non-empty subset of a Banach space* E. *For a mapping* $T : C \to E$ *the following are equivalent*:

(1) T *is pseudo-lipschitzian with constant* k.

(2) $$\lim_{r \to 0+} (|x - y - r(Tx - Ty)| - |x - y|)/(-r) \leq k|x - y|$$
for all x and y in C.

(3) *For each x and y in* C, *there is $j \in J(x - y)$ such that* $(Tx - Ty, j) \leq k|x - y|^2$.

If C is a convex subset of E and x is in C, we define $I(C, x) = \{z \in E : z = x + a(y - x)$ for some $y \in C$ and $a \geq 0\}$. The closure $\mathrm{cl}(I(C, x))$ of this subset is sometimes called the support cone to C at x [8, p. 23]. A mapping $T : C \to E$ is said to be weakly inward [7, p. 353] if $Tx \in \mathrm{cl}(I(C, x))$ for each x in C. For z in E we set $d(z, C) = \inf\{|z - y| : y \in C\}$. A functional $x^* \neq 0$ in E^* is said to support C at $x \in C$ if $(x, x^*) = \max\{(y, x^*) : y \in C\}$.

Our next lemma is also essentially known [12, p. 62, 3, 13, and 5, p. 368]. It yields Proposition 2.

LEMMA 2. *Let* C *be a convex subset of* E. *For z in* E *and x in* C, *the following are equivalent*:

(1) $$z \in \mathrm{cl}(I(C, x)).$$

(2) $$\lim_{h \to 0+} d((1 - h)x + hz, C)/h = 0.$$

(3) *If $x^* \in E^*$ supports* C *at x, then* $(z, x^*) \leq (x, x^*)$.

Proof. (1) $\Rightarrow$ (3) is immediate. If z does not belong to cl (I (C , x)), then there exists x^* in E^* such that $(z, x^*) > \sup \{(y, x^*) : y \in \text{cl}(I(C, x))\}$. The functional x^* must support cl (I (C , x)) at x. Consequently, (3) $\Rightarrow$ (1). (2) $\Rightarrow$ (1) is also immediate. To prove (1) $\Rightarrow$ (2), let $\varepsilon > 0$ be given. There are $a > 0$ and $y \in C$ such that $|z - (x + a(y - x))| < \varepsilon$. Let $0 < h < 1/a$. Then $d((1-h)x + hz, C)/h \leq |(1-h)x + hz - ((1-ah)x + ahy)|/h < \varepsilon$.

PROPOSITION 2. *Let* C *be a convex subset of* E. *For a mapping* $T : C \to E$, *the following are equivalent*:

(1) T *is weakly inward.*

(2) $$\lim_{h \to 0+} d((1-h)x + hTx, C)/h = 0 \quad \textit{for each} \quad x \in C.$$

(3) *If* $x^* \in E^*$ *supports* C *at* x, *then* $(Tx, x^*) \leq (x, x^*)$.

We remark in passing that by Proposition 2 T satisfies Bony's condition [1] in Hilbert space if and only if it is weakly inward (cf. [4] and [13]).

We shall denote the family of continuous $T : C \to E$ which are both weakly inward and pseudo-lipschitzian with constant k by W (C , k). In order to investigate $\inf\{|x - Tx| : x \in C\}$ for T in W (C , k) we define, for each E and k, $p(E, k) = \sup\{\inf\{|x - Tx| : x \in C\}/R(C) : C \in B(E)$ and $T \in W(C, k)\}$.

2. MAIN RESULTS

If $k < 1$, then $p(E, k) = 0$ for all E. This follows from [11, p. 413]. The Halpern-Bergman fixed point theorem [7, p. 356] shows that $p(E, k) = 0$ for all k provided E is finite-dimensional. Therefore we shall assume in the sequel that E is infinite-dimensional and that $k \geq 1$. If E is fixed (but arbitrary), we shall write $p(k)$ (and $g(k)$) instead of $p(E, k)$ (and $g(E, k)$).

THEOREM 1. $p(k) \leq k - 1$ *for* $k \geq 1$.

Proof. Let ε be positive, and let $0 < t < 1/k$. Let $y \in C$ satisfy $\sup\{|y - x| : x \in C\} < R(C) + \varepsilon$, and define $S : C \to E$ by $Sx = (1-t)y + tTx$ for each x in C. S belongs to W (C , tk). Therefore it has a (unique) fixed point z [11, p. 413]. Since $|z - Tz| = (1-t)|y - z|/t < (1-t)(R(C) + \varepsilon)/t$, the result follows.

Goebel's examples [6, p. 154] show that in some cases the upper bound obtained in Theorem 1 is exact.

A lemma will precede the derivation of a lower bound for $p(k)$.

LEMMA 3. *The right derivative* $p'_+(1)$ *exists.*

Proof. Suppose that T belongs to $W(C, k)$ and that $1 < m \leq k$. Let $t = (m-1)/(k-1)$ and define $S: C \to E$ by $S = (1-t)I + tT$. A simple computation shows that S belongs to $W(C, m)$. Since $|x - Sx| = t|x - Tx|$, we obtain $p(m) \geq tp(k)$. In other words, $p(m)/(m-1) \geq p(k)/(k-1)$. This yields our assertion.

Observe that $p(k) = 0$ for some (hence all) $k > 1$ if and only if $p'_+(1) = 0$.

THEOREM 2. $p'_+(1)(1 - 1/k) \leq p(k)$ *for* $k \geq 1$.

Proof. Let $1 < m < k$, $T \in W(C, m)$, and $0 < t < 1/m$. For each z in C, define $S: C \to E$ by $Sx = (1-t)z + tTx$. S belongs to $W(C, tm)$ and has a unique fixed point Fz. Consider the mapping $B: C \to E$ defined by $B = TF$. B is weakly inward (in fact, it is also inward in the sense of [7]) because $Bz = z + (Fz - z)/t$. Now let x and y be in C and choose $j \in J(Fx - Fy)$ such that $(TFx - TFy, j) \leq m|Fx - Fy|^2$. We have $|Fx - Fy|^2 = (Fx - Fy, j) = (1-t)(x-y, j) + t(TFx - TFy, j) \leq (1-t)|x-y||Fx - Fy| + tm|Fx - Fy|^2$. Consequently, $|Fx - Fy| \leq (1-t)|x-y|/(1-mt)$ and if $j \in J(x-y)$ then $(Bx - By, j) = (1 - 1/t)|x-y|^2 + (Fx - Fy, j)/t \leq (1-1/t)|x-y|^2 + |Fx - Fy||x-y|/t \leq m(1-t)|x-y|^2/(1-mt)$. Thus B is continuos and belongs to $W(C, m(1-t)/(1-mt))$. Therefore $|Fx - TFx| = (1-t)|x - Bx|$ implies that $p(m) \leq (1-t)p(m(1-t)/(1-mt))$. Setting $t = (k-m)/(m(k-1))$ we obtain $mp(m)/(m-1) \leq kp(k)/(k-1)$. Since $1 < m < k$ was arbitrary, the result follows by letting m tend to 1.

We now return to Goebel's function $g(k)$. It is clear that $g(k) \leq p(k)$. Here is a result in the other direction.

THEOREM 3. $p(k) \leq (k-1)g'_+(1) \leq kg(k)$ *for* $k \geq 1$.

Proof. Let T be in $W(C, k)$. For each positive r such that $r(k-1) < 1$, the image of C under the mapping $I + r(I - T)$ contains C. Therefore one may define a mapping $J_r: C \to C$ by $J_r = [I + r(I-T)]^{-1}$. This mapping is lipschitzian with constant $t = 1/(1 - r(k-1))$. Therefore it belongs to $L(C, t)$ and for each positive ε there exists x in C such that $|x - J_r x| \leq (g(t) + r\varepsilon)R(C)$. We have $|J_r x - TJ_r x| = |x - J_r x|/r \leq (g(t)/r + \varepsilon)R(C)$. Consequently, $p(k) \leq g(t)/r = (k-1)tg(t)/(t-1)$. This means that $p(k) \leq (k-1)g'_+(1)$. This completes the proof because $g'_+(1) \leq kg(k)/(k-1)$.

3. SOME REMARKS

The bounds established in Theorem 3 enable us to see that $p(k) = 0$ for $k > 1$ if and only if $g(k) = 0$. (In fact, it can be shown that $p'_+(1) = g'_+(1)$). Thus our results seem to suggest that sometimes $p = g$. We do not know when this is indeed true.

Let H be a Hilbert space. By [6, p. 160] $g'_+(\mathrm{H}, 1) \leq 1/\sqrt{2}$. Therefore $p(\mathrm{H}, k) \leq (k-1)/\sqrt{2}$. This bound is smaller than the bound obtained in Theorem 1. If $p(\mathrm{H}, k) > 0$ for $k > 1$ (by [6, p. 161] this is the case if and only if there exists a lipschitzian retraction of the unit ball of H onto its boundary), then $\lim_{k\to\infty} p(k) = \infty$. Therefore the lower bound in Theorem 2 is also not exact if $p(\mathrm{H}, k) > 0$.

Added in proof. In a supplementary note it is shown that $p(k) = (k-1)g'_+(1)$. Consequently, $p(k) = g(k)$ if and only if $g(k) = 0$.

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