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Eigenvalues of densifying mappings

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Analisi funzionale. — *Eigenvalues of densifying mappings.* Nota di KANHAYA LAL SINGH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono stabiliti risultati sugli autovalori degli operatori non lineari negli spazi di Banach e Hilbert detti « densifying mappings ».

1.1. Nonlinear operator equations in Banach spaces and Hilbert spaces have been studied in recent years by several mathematicians. The aim of the present paper is to prove some results about the eigenvalues of densifying mappings.

The notion of measure of noncompactness was introduced by C. Kuratowskii [1]. Let X be a real Banach space and D be a bounded subset of X. The measure of noncompactness of D, denoted by $\gamma(D)$ is defined as follows

$\gamma(D) = \inf\{\varepsilon > 0/D$ can be covered by a finite number of subsets of diameter $< \varepsilon\}$.

$\gamma(D)$ has the following properties:

(1) $0 \leq \gamma(D) < d(D)$, where $d(D)$ denotes the diameter of D,

(2) $\gamma(D) = 0$ if and only if D is precompact,

(3) $\gamma[C \cup D] = \max\{\gamma(C), \gamma(D)\}$,

(4) $\gamma(C(D, \varepsilon)) < (D) + 2\varepsilon$, where $C(D, \varepsilon) = \{x \text{ in } X/d(x, D) < \varepsilon\}$,

(5) $C \subset D$ implies $\gamma(C) \leq \gamma(D)$,

(6) $\gamma(C + D) < \gamma(C) + \gamma(D)$, where $C + D = \{c + d/c \in C, d \in D\}$.

Closely related to the notion of measure of noncompactness is the concept of k-set contraction defined by Darbo [2] as follows:

DEFINITION 1.1. Let X be a Banach space. Let D be a bounded subset of X. Let $T: D \rightarrow X$ be continuous. T is said to be *k-set contraction* if $\gamma(T(D)) < k\gamma(D)$ for some $k > 0$. If $k < 1$, i.e.

$$\gamma(T(D)) < \gamma(D)$$

T is called densifying.

The degree theory for densifying mapping was introduced by Nussbaum [3] and Sadovskii [4], although Sadovskii's measure of noncompactness is not the same as that of Kuratowskii's, but yet they share few properties in common.

(*) Nella seduta dell'11 giugno 1975.

DEFINITION 1.2. A densifying mapping $T : D \to X$ is said to be *positive* if it transforms positive vectors into positive vectors.

THEOREM 1.1. *Let* D *be open, bounded subset of a Banach space* X *such that origin belongs to* D. *Let* $F : D \to X$ *be densifying. Suppose* T *does not have a fixed point* D. *Then there exists* $\lambda_1 > 0$ *and* $\lambda_2 < 0$ *and* x_1, x_2 *such that* $F(x_j) = \lambda_j x_j, j = 1, 2$, *i.e.* F *has a negativeeigen value and a positive (nonnegative) eigenvalue.*

Proof. In the proof of Theorem 1.1 we will make use of the degree theory for densifying mappings. Consider the homotopy $H(x, t) : \bar{D}x [0, 1] \to X$ defined by

$$H(x, t) = (1 - t) F(x), x \text{ in } D, t \text{ in } [0, 1].$$

Then $H(x, t)$ is densifying. Clearly $H(x, t)$ is continuous. Let A be any bounded but not precompact subset of D, then by definition of $H(x, t)$ we have

$$H(A, t) = (1 - t) F(A).$$

Hence $\gamma(H(A, t)) = \gamma((1 - t) F(A)) \leq \gamma(1 - t) A < \gamma(A)$.

Now we claim that $H(x, t)$ is uniformly continuous in t for t in $[0, 1]$. Indeed, let $| t - s | \delta/M$; we need to show that $\| H(x, t) - H(x, s) \| < \varepsilon$. Now by definition of $H(x, t)$ we have

$$\| H(x, t) - H(x, s) \| = \| (1 - t) F(x) - (1 - s) F(x) \| =$$

$$= \| (s - t) F(x) \| = \| (t - s) F(x) \| \leq | (t - s) | M$$

where $M \geq \| F(x) \|$. Thus taking $\varepsilon = \delta$ we get

$$\| H(x, t) - H(x, s) \| < \varepsilon.$$

Thus $H(x, t)$ is a well defined homotopy.

Finally we claim that $x - H(x, t) = 0$, for all x in ∂D and t in $[0, 1]$. Suppose the contrary, i.e. $x - H(x, t) \neq 0$ for some x in D and t in $[0, 1]$. Then

$$\mathrm{Deg}(I - H(\cdot, 0), D, O) = \mathrm{Deg}(I - H(\cdot, 1), D, O).$$

But $\mathrm{Deg}(I - H(\cdot, 0), D, O) = \mathrm{Deg}(I, D, O)$, and

$$\mathrm{Deg}(I - H(\cdot, 1), D, O) = \mathrm{Deg}(I - F, D, O).$$

Hence $\mathrm{Deg}(I - F, D, O) = \mathrm{Deg}(I, D, O) \neq 0$.

Thus there exists a x in $\bar{D}$ such that $x - F(x) = 0$, i.e. $F(x) = x$, a contradiction to the hypothesis that F does not have a fixed point.

Thus we conclude that $x - H(x, t) = x - (1 - t) F(x) = 0$ for all x in D and t in $[0, 1]$. Now we have following three cases.

Case 1. If $t = 0$, then $x - F(x) = 0$, which implies $F(x) = x$, i.e. F has a fixed point, a contradiction to the hypothesis.

Case 2. If $t = 1$, $x = 0$, which in turns implies that x is an interior point, a contradiction to the fact that x is boundary point.

Case 3. If, $0 < t < 1$, then $x_1 - (1-t) F(x_1) = 0$ implies $x_1 = (1-t) F(x_1)$, which in turns implies $F(x_1) = \frac{x_1}{1-t}$.

Let $\lambda_1 = \frac{1}{t(1/t - 1)}$, then clearly $\lambda_1 > 0$ and $F(x_1) = \lambda_1 x_1$. Thus the first half of the theorem is proved.

For the second half, we define the homoyopy $H(x, t)$; $\bar{D}x [0, 1] \to X$ as follows

$$H(x, t) = 2(1-t)x - (1-t)F(x), x \text{ in } \bar{D}, t \text{ in } I, \quad \text{where} \quad I = [0, 1].$$

Clearly $H(x, t)$ is densifying. Indeed, let A be any bounded but not precompact subset of D, then

$$H(A, t) = 2(1-t)A - (1-t)F(A).$$

Hence
$$\begin{aligned} \gamma(H(A, t)) &= \gamma(2(1-t)(A)) - \gamma(1-t)F(A) \leq \\ &\leq 2(1-t)\gamma(A) - (1-t)\gamma(A) = \\ &= (2 - 2t - 1 + t)\gamma(A) = (1-t)\gamma(A) < \gamma(A). \end{aligned}$$

As in the case of first part it can be easily shown that $H(x, t)$ is uniformly continuous in t for t in I.

Now we claim that $x - H(x, t) = (2t-1)x + (1-t)F(x) = 0$ for all x in $\bar{D}$ and for all t in I. Suppose the contrary, i.e. $x - H(x, t) \neq 0$. Then by the homotopy theorem $\text{Deg}(I - H(x, t), D, O)$ is constant in t for t in I. Thus

$$\text{Deg}(I - H(\cdot, O), D, o) = \text{Deg}(I - H(\cdot, 1), D, O).$$

But
$$\text{Deg}(I - H(\cdot, o), D, o) = \text{Deg}(-(I - F), D, O)$$

and
$$\text{Deg}(I - H(\cdot, 1), D, O) = \text{Deg}(I, D, O) \neq 0.$$

Therefore there exists an x in D such that $-(I - F)x = 0$, i.e. $F(x) = x$, a contradiction to the hypothesis that F does not have a fixed point.

Thus we conclude that $x - H(x, t) = (2t-1)x + (1-t)F(x) = 0$. Now we have the following three cases:

Case 1. If $t = 0$, then $(2t-1)x + (1-t)F(x) = 0$ implies that $-(I - F)x = 0$, which in turns implies that $F(x) = x$, a contradiction to the hypothesis.

Case 2. If $t = 1$, then $(2t-1)x + (1-t)F(x) = 0$ implies that $x = 0$, i.e. x is an interior point, a contradiction to the fact that x is a boundary point.

Case 3. If $0 < t < 1$, then $(2t - 1)x_2 + (1 - t)F(x_2) = 0$ implies that $(2t - 1)x_2 = -(1 - t)F(x_2)$, which in turns implies that

$$F(x_2) = \frac{(2t - 1)x_2}{t - 1} = \frac{(2 - 1/t)x_2}{(1 - 1/t)} .$$

Let $\lambda_2 = \frac{2 - 1/t}{1 - 1/t}$, then clearly $\lambda_2 < 0$ and $F(x_2) = \lambda_2 x_2$. Thus the Theorem is proved.

Now we prove a theorem on the eigenvalues of positive densifying mappings.

DEFINITION 2.1. Let X be a Banach space. Let D be a subset of X. D is said to be convex, if for any two points x, y in D, the line segment joining x and y also belongs to D, i.e.

$$ax + (1 - a)y \in D ,$$

where $0 \leq a \leq 1$.

The closed, convex subset D of X is said to be a cone if the following conditions are satisfied:

(*a*) If x in D, then tx in D for all $t \geq 0$;

(*b*) If $x \neq 0$, then at least one of the vectors (points) $x, -x$ in X does not belong to D.

Notation. We will denote by D_r the set of positive vectors with norm not exceeding r. Clearly the set D_r is convex, since it is intersection of two convex sets, namely the cone D and the ball B_r of vectors with norm not exceeding r.

THEOREM 2.1. *Let* T *be a positive densifying operator such that for some* $r_0 > 0$

$$d(\theta, T(D_{r_0})) = \inf_{xD} \| Tx \| > 0 .$$

Then the operator T *has at least one eigenvector* x_0, $\| x_0 \| = r_0$ *in the cone* D, *corresponding to a positive eigenvalue* q, $T(x_0) = qx_0$.

Proof. Define the operator $\hat{T}$ on D_{r_0} by the equation

$$\hat{T}(x) = \begin{cases} \dfrac{r_0 T(x)}{\| T(x) \|} & \text{if } x \text{ in D} \\ 0 & \text{otherwise} . \end{cases}$$

Then clearly T is densifying. Indeed, taking $f_1(x) = T(x)$, $f_2(x) = 0$ $a_1(x) = r_0$ if x in D_{r_0} and $a_2(x) = 1 - a_1(x)$ we conclude by Theorem (9, pp. 17, 3) that T is densifying. Moreover T maps D_{r_0} into itself. Since, for x in D_{r_0}

$$\| \hat{T}(x) \| = \left\| \frac{r_0 T(x)}{\| T(x) \|} \right\| = r_0 .$$

Hence by Darbo's Theorem there exists a point x_0 in D_{r_0} such that $\hat{T}(x_0) = x_0$. Hence

$$\hat{T}(x_0) = \frac{r_0 T(x_0)}{\| T(x_0) \|} = x_0 \tag{2}$$

or

$$T(x_0) = \frac{x_0 \| T(x_0) \|}{r_0} .$$

Let $q = \frac{\| T(x_0) \|}{r_0}$, then clearly $q > 0$ and equation (1) is satisfied with q, and clearly from (2) we have

$$\| x_0 \| = \left\| \frac{r_0 T(x_0)}{\| T(x_0) \|} \right\| = \frac{r_0 \| T(x_0) \|}{\| T(x_0) \|} = r_0 .$$

Hence the Theorem.

As a corollary of Theorem 2.1 we have the following corollary, which is a generalization of a Theorem of E. Rothe [5].

COROLLARY 2.1. *Let* X *be a Banach space. Let* D *be a cone in* X. *Let* $T : D \to X$ *be a positive densifying mapping satisfying*

$$\inf_{x \in D, \|x\| = r_0} \| T(x) \| > 0 .$$

Then T *has an eigenvector* x_0, *moreover* $\| x_0 \| = r_0$.

Proof. Define the mapping $F(x)$ on D_{r_0} (where D_{r_0} has the same meaning as in the paragraph preceding to Theorem 2.1) by

$$F(x) = \| x \| \frac{T(r_0 x)}{\| x \|} + (r_0 - \| x \|) u \qquad (x \in D_{r_0}) ,$$

where u is some fixed element of the cone D. Then clearly (in fact this can be seen by a direct application of a Lemma due to Porter) $F(x)$ is a k-set contraction with $k < 1$, moreover for $F(x)$ the condition of Theorem 2.1

$$d(\theta , F(D_{r_0})) = \inf_{x \in D r_0} \| F x \| > 0$$

is satisfied. Hence for x in D, $\| x \| = r_0$ and $F(x) = r_0 T(x)$. Indeed, $F(x) = x = \| x \| \left(\frac{T(r_0 x)}{\| x \|} \right) + (r_0 - \| x \|) u$. But for x in D we have $\| x \| = r_0$, hence

$$F(x) = r_0 T(x) .$$

Remark 2.1. Since every completely continuous mapping is densifying, as a corollary of Corollary 2.1 we have the following Corollary due to E. Rothe [5].

COROLLARY 2.2. *A positive completely continuous operator* T *has an eigenvalue* x_0, $\| x_0 \| = r_0$ *if the following condition is satisfied*

$$\inf_{x \in D, \| x \| = r_0} \| T(x) \| > 0 .$$

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