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An approximation method of the first eigenvalue in nonlinear eigenvalue problems

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Analisi funzionale. — *An approximation method of the first eigenvalue in nonlinear eigenvalue problems.* Nota di ANNA MARIA MICHELETTI e FRANCESCO ZIRILLI presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Sia $\{\psi_i\}_{i=1}^{\infty}$ una base ortonormale completa di uno spazio di Hilbert separabile H, e $V^N \subset H$ il sottospazio generato da $\{\psi_1, \psi_2 \cdots \psi_N\}$. Sia $\Phi(u)$ $u \in H$ un funzionale pari limitato inferiormente di classe $\mathscr{C}^2$ e $\tilde{S} = \{v \in H \mid \|v\| = 1$ con i punti antipodali identificati$\}$. Si dimostra che, se $(\tilde{S}, \Phi)$ soddisfa la condizione di Palais–Smale, allora $c_1^N(\Phi) \to c_1, (\Phi)$ quando $N \to +\infty$, dove

$$c_1^N(\Phi) = \inf_{\substack{\text{cat}(A) \geq 1 \\ A \subset \tilde{S}}} \sup \{\varphi(u) \mid u \in A \cap V^N\} \quad \text{e} \quad c_1(\psi) = \inf_{\substack{\text{cat}(A) \geq 1 \\ A \subset \tilde{S}}} \sup \{\varphi(u) \mid u \in A\}.$$

We recall firstly some definitions.

DEFINITION 1. Let Y be a topological space and A a closed subset of Y. The category cat (A) = cat (A , Y) of A relative to Y is the least integer k for which there exist k closed sets $A_1 \cdots A_k$ such that $\bigcup_{j=1}^{k} A_j \supseteq A$ and such that the identity map $i : A_j \to Y$ is homotopic to a constant; if no such integer exists we write cat (A) $= +\infty$ [3].

DEFINITION 2. Let Y be a topological space and f a continuous function on Y. Let $m \geq 1$ be an integer. Set

$$c_m(f) = \inf_{\text{cat}(A) \geq m} [\sup \{f(p) \mid p \in A\}] \qquad m \leq \text{cat}\, N.$$

Trivially $c_1(f) \leq c_2(f) \leq \cdots c_n(f) \leq c_{n+1}(f) \leq \cdots$ [3].

Palais-Smale (P — S) *Condition.* Let M be a complete C^2 Riemannian manifold without boundary modeled on a separable Hilbert space. And $f : M \to R$ be a C^2 functional. The pair (M, f) satisfies the P—S condition if $\{p_n\} \subseteq M$ is a sequence of points such that $f(p_n)$ is bounded and $(\nabla f)(p_n)$ converges to zero implies that p_n contains a convergent subsequence [3].

The main result of the Lusternik-Schirelman theory of critical points is the following.

THEOREM 1. *Let* (M, f) *satisfy the Palais-Smale* (P — S) *condition and let* f *be bounded below and let* $c_m(f)$ *be as in Def.* 2. *Then* $c_m(f)$ *are critical values of* f *with critical vectors* u_m (i.e. $f(u_m) = c_m(f)$ $(\nabla f)\, u_m = 0$), *and* f *has at least cat* (M) *critical points* [2].

(*) Nella seduta del 10 maggio 1975.

Let H be a separable Hilbert space and $\Phi(u) : H \to R$ a C^2 even bounded below functional; then the nonlinear eigenvalue problem

$$\nabla \Phi u \equiv T u = \lambda u \,, \qquad \| u \| = 1$$

is equivalent to finding the critical points of $\Phi(u)$ on the manifold $\tilde{S} = \{ x \in H \mid \| x \| = 1 \text{ with the antipodal points identified} \}$. It has been shown that cat $\tilde{S} = +\infty$ [1].

Suppose $(\tilde{S}, \Phi)$ satisfies the P — S condition. Then, from Theorem 1, $c_1(\Phi) = \inf_{\text{cat}(A) \geq 1} \sup [\varphi(p) \mid p \in A] = c_1$ is the minimum of $\Phi(\cdot)$. There exists u_1 such that

$$\nabla \varphi u_1 \equiv T u_1 = c_1 u_1 \,, \qquad \| u_1 \| = 1 .$$

Let $\{\psi_i\}_1^\infty$ be a complete orthonormal basis of H and

$$u_1 = \sum_1^\infty a_i \psi_i \quad , \quad V^N = [\psi_1 \cdots \psi_N\}.$$

We shall establish

THEOREM 2. *Under the hypothesis of Theorem 1,*

$$c_1^N = \inf_{\text{cat} A \geq 1} \sup \{ \Phi(u) \mid u \in A \cap V^N \}$$

converge to c_1 *when* $N \to +\infty$.

Proof. It is obvious that

$$c_1(\Phi) \leq c_1^N(\Phi) \leq \Phi(u_1^N)$$

where $u_1^N = \sum_1^N a_i \psi_i$ but $\Phi(u_1^N) \to \Phi(u_1) = c_1$ when $N \to +\infty$ so $c_1^N \to c_1$.

Q.E.D.

Some comments. Many problems in mathematical physics and engineering can be understood as nonlinear variational eigenvalue problems subjected to constraints. If one is interested in having an explicit value of the first eigenvalue, the only thing that can be done on a computer is to solve the problem on some $V^N = [\psi_1 \cdots \psi_N]$ that means compute c_1^N. Under the hypothesis previously given we show $c_1^N \to c_1$.

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