Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Orthogonality Preserving Operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.1-2, p. 26–31.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_1-2_26_0>

Analisi funzionale. — *Orthogonality Preserving Operators* Nota II (*).
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RIASSUNTO. — Richiamandosi alla Nota I in cui si riprende un problema considerato da S. Pincherle (1928), da W. Hahn (1949) e successivamente da altri, i due Autori trovano classi di polinomi ortogonali per le quali esiste una trasformazione preservante l'ortogonalità.

1. INTRODUCTION

In this note we continue our previous note (we refer to it as (1)). We consider this time the linear operator defined on polynomials by means of

$$\mathrm{L}\{x^n\} = \mu_n x^{n-1} \qquad (n = 0, 1, 2 \cdots), \tag{1.1}$$

where $\mu_0 = 0$, $\mu_n = n\lambda_{n-1} \neq 0$ $(n = 1, 2, 3, \cdots)$. This opeator can be written as

$$\mathrm{L} = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \mathrm{D}^{n+1} = \sum_{n=0}^{\infty} \frac{b_n}{n!} (x\mathrm{D})^n \mathrm{D} \tag{1.2}$$

so that

$$\lambda_n = \sum_{k=0}^{n} \binom{n}{k} a_k .$$

As in (I) we raise the question *for what operators* L *of the type* (1.1) *and for what* OPS $\{p_n(x)\}$ *is the set* $\{q_n(x) = \mathrm{L}p_{n+1}(x)\}$ *also an* OPS?

Hahn [3] posed and solved this problem for $\mu_n = n$ (L = D). In this respect our problem is a generalization of Hahn's. Krall and Sheffer [4] considered the problem of finding OPS such that

$$Q_n(x) = \sum_{j=0}^{k} a_j(x)\, p_{p+1}^{(j+1)}(x)$$

is also an OPS. They gave necessary and sufficient conditions so that this will happen. They gave several, but not exhaustive, examples of such cases.

In this note we give necessary and sufficient conditions and solve the problem completely in case the OPS is symmetric. We are however unable to solve completely the non-symmetric case. We only give examples.

(*) These « Rend. Lincei, Sc. fis. mat. e nat. », *58* (6), 833–838.
(**) Nella seduta del 10 maggio 1975.

2. NECESSARY AND SUFFICIENT CONDITIONS

We give now the following

THEOREM I. *Let* $\{p_n(x)\}$ *be an* OPS *with associated moments* $\{\alpha_n\}$. *In order that* $\{q_n(x) = \mathrm{L}p_{n+1}(x)\}$ *where* L *is of type* (I.I) *be also an* OPS *it is necessary and sufficient that there exist constants* $\{\alpha_n^*\}$ *and* $\{\mathrm{A}_{ji}^*\}$ *such that*

(2.1) $\lambda_n \neq 0 \quad (n = 0, 1, 2, \cdots)$; (2.2) $\mathrm{A}_{s+1,s+1}^* \neq 0$

$$\sum_{k=0}^{s=1} \mathrm{A}_{s+1,k}^* \alpha_{n+k} = n\lambda_{n-1} \alpha_{n+s-1}^* \qquad (n, s = 0, 1, 2, \cdots) \tag{2.3}$$

when these conditions are satisfied $\{\alpha_n^*\}$ *are the moments associated with* $\{q_n(x)\}$.

The proof of this theorem is similar to that given by Krall and Sheffer and we omit it.

3. THE SYMMETRIC CASE

We first note that putting $s = 0$ in (2.3) and using the fact that the odd moments are zero, we get $\mathrm{A}_{10}^* = 0$ and we can take $\mathrm{A}_{11}^* = 1$. Thus we can rewrite (2.3) as

$$\sum_{k=0}^{s+1} \mathrm{A}_{s+1,k}^* \alpha_{n+k+1} = \frac{\mu_{n+1}}{\mu_{n+s+1}} \alpha_{n+s+2} \qquad (n, s = 0, 1, 2, 3, \cdots). \tag{3.1}$$

Now put $s = 1, 2, 3$ in (3.1) and replacing n by $2n - 1$, $2n$ and $2n - 1$ respectively and putting $\xi_n = \alpha_{2n}/\alpha_{2n+2}$ we get

$$\mathrm{A}_{20}^* \xi_n + \mathrm{A}_{22} = \frac{\mu_{2n}}{\mu_{2n+1}} \tag{3.2}$$

$$\mathrm{A}_{31} \xi_n + \mathrm{A}_{33} = \mu_{2n-1}/\mu_{2n+1} \tag{3.3}$$

and

$$\mathrm{A}_{40}^* \xi_n \xi_{n+1} + \mathrm{A}_{42}^* \xi_n + \mathrm{A}_{44}^* = \mu_{2n}/\mu_{2n+3}. \tag{3.4}$$

Substituting for (μ_{2n}/μ_{2n+3}) in (3.4) from (3.2) and (3.3) we get that $\{\xi_n\}$ must satisfy a recurrence relation of the form

$$\mathrm{C}\xi_n \xi_{n+1} + \mathrm{B}\xi_n + \mathrm{A}\xi_{n+1} = \mathrm{D}. \tag{3.5}$$

As in (I) this has the solution

$$\xi_n = \alpha_{2n}/\alpha_{2n+2} = \frac{1}{a} \frac{1 - \beta q^n}{1 - \alpha q^n}, \quad \text{i.e.,} \quad \alpha_{2n} = a^n \frac{[\alpha]_n}{[\beta]_n}. \tag{3.6}$$

Again we have disregarded solutions of the form $\xi_n =$ constant for all n or for $n \geq \mathrm{N}$ or for $n \leq \mathrm{N}$ as these would lead to Hankel determinants for the

moments which vanish, in contradiction to well known necessay conditions for $\{a_n\}$ to be a moment sequence.
Thus we have proved the theorem

THEOREM 2. *Let* $\{p_n(x)\}$ *be a symmetric* OPS *such that there exists an operator of the form* (1.1) *with the property that* $\{Lp_{n+1}(x) = q_n(x)\}$ *is also an* OPS. *Then we have*

$$p_{2n}(x) = J_n(q, \alpha, \beta; x^2/a) \quad , \quad p_{2n+1}(x) = x J_n(q, \alpha q, \beta q; x^2/a) \tag{3.7}$$

whereas the transformed set is

$$\begin{aligned} q_{2n}(x) &= \frac{[\alpha q]_n}{[\delta q]_n} J_n(q, \delta q\, \beta q; dx^2/a) \\ q_{2n+1}(x) &= ed \frac{(1-q^{n+1})[\alpha q]_n}{[\delta q]_{n+1}} x J_n(q, \delta s^2, \beta q^2; dx^2/a) \end{aligned} \tag{3.8}$$

and the operator L *is given by*

$$\begin{aligned} Lx^{2n} &= e\, d^n (1-q^n) \frac{[\alpha q]_{n-1}}{[\delta q]_n} x^{2n-1} \\ Lx^{2n+1} &= d^n \frac{[\alpha q]_n}{[\delta q]_n} x^{2n} \end{aligned} \tag{3.9}$$

We can verify the orthogonality of $\{p_n(x)\}$ and of $\{q_n(x)\}$ in the above theorem using the same method that was used at the end of § 3 in (I). However we make instead the following observation. First recall that $S_n(x) = J_n(q, \alpha, \beta; x/a)\, a^n$ are orthogonal on $(0, \infty)$ with respect to $d\psi(x)$ (see I). Secondly the polynomials $Q_n(x) = a^n J_n(q, \alpha q, \beta q; x/a)$ are the kernel polynomials $K_n(0, x)$ given by

$$K_n(0, x) = \frac{1}{x}\left\{S_{n+1}(x) - \frac{S_{n+1}(0)}{S_n(0)} S_n(x)\right\}$$

which are orthogonal on $(0, \infty)$ with respect to $x d\psi(x)$. Hence the polynomial set $\{p_n(x)\}$ of (3.7) are those symmetric OPS constructed by means of [2] $p_{2n}(x) = S_n(x^2)$ and $p_{2n+1}(x) = xQ_n(x^2)$. Similar remarks apply to (3.8).

4. SPECIAL CASES

We mention now some interesting special cases for the results of our note (I) and of our present note.

First recall the results of (I–§ 4). If we replace α by $q^{\alpha+1}$ β by $q^{\alpha+\beta+2}$ and γ by $q^{\gamma+1}$ and then let $q \to 1$ we obtain the special result that

$$J = \sum_0^\infty \frac{(-1)^k}{k!}\, {}_2F_1(-k, \alpha+1; \gamma+1; b)\, x^k D^k \tag{4.1}$$

takes the polynomial set $P_n^{(\alpha,\beta)}(1-2x)$ into the set

$$\frac{(\alpha+1)_n}{(\gamma+1)_n} P_n^{(\gamma,\alpha+\beta-\gamma)}(1-2bx).$$

To see this we only need to observe that as $q \to 1$

$$J_n(q, q^{\alpha+1}, q^{\alpha+\beta+2}; x) \to \frac{n!}{(\alpha+\beta+n+1)_n} P_n^{(\alpha,\beta)}(1-2x).$$

If we specialize further by putting $b=1$ in (4.1) we see that the operator of

$$J = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\gamma-\alpha)_n}{(\gamma+1)_n} x^n D^n = {}_2F_1(-xD, \gamma-\alpha; \gamma+1; 1) = \tag{4.2}$$

$$= \frac{\Gamma(\gamma+1)\,\Gamma(\alpha+1+xD)}{\Gamma(\alpha+1)\,\Gamma(\gamma+1+xD)}$$

takes the polynomial set $p_n^{(\alpha,\beta)}(1-2x)$ to the polynomial set

$$\frac{(\alpha+1)_n}{(\gamma+1)_n} P_n^{(\gamma,\alpha+\beta-\gamma)}(1-2x).$$

Specializing still further by putting $\alpha = \gamma + N$ where N is a positive integer we get that the finite operator

$$J = \frac{(\gamma+1+xD)_N}{(\gamma+1)_N} \tag{4.3}$$

takes the Jacobi polynomial $p_n^{(\gamma+N,\beta)}(1-2x)$ to the set

$$\frac{(\gamma+N+1)_n}{(\gamma+1)_n} P_n^{(\gamma,\beta+N)}(1-2x).$$

This case appeared in [4, p. 440].

If we now go back to the operator (4.1) replace b by b/α and then let $\alpha \to \infty$ we see that the operator

$$J = \sum_{n=0}^{\infty} \frac{(-1)_n}{(\gamma+1)_n} L_n^{(\gamma)}(b)\, x^n D^n \tag{4.4}$$

takes the Bessel polynomials (for notation and references see [1]) $Y_n^{(\beta)}(x)$

$$J : Y_n^{(\beta)}(x) \to \frac{n!}{(\gamma+1)_n} P_n^{(\gamma,\beta-\gamma)}(1-2bx).$$

In a similar fashion one can see that if

$$J = \sum_{p=0}^{\infty} \frac{(-1)^n}{n!} {}_2F_1(-n, c+1; \delta+1; b)\, x^n D^n \tag{4.5}$$

then

$$J : L_n^{(c)}(x) \to \frac{(c+1)_n}{(\delta+1)_n} L_n^{(\delta)}(bx).$$

If we now recall the results of § 3 we first see that these results cover that part of Hahn's theorem for symmetric orthogonal polynomial sets.

Indeed if we put in formulas (3.9) $\alpha = q^{1/2}$, $\delta = q^{-1/2}$, $d = e = 1$, and $a = 1/1 - q$ and then take the limits as $q \to 1$ we get that $\mu_n = n$, $n = 0, 1, 2, \cdots$ so that $L = D$. If in addition to the above choices of the constants we put $\beta = 0$ we get that $p_n(x) = \widehat{H}_n(x)$ the monic Hermite polynomials.

On the otherh and had we replaced β by $q^{\beta+1}$ before taking the limit as $q \to 1$ we get from (3.7) that

$$p_{2n}(x) = \frac{(1/2)_n}{(\beta + n)} \, {}_2F_1\left(-n, n + \beta; \tfrac{1}{2}; x^2\right)$$

and

$$p_{2n+1}(x) = \frac{\left(\frac{3}{2}\right)_n}{(\beta + n + 1)_n} \, xF\left(-n + \beta + 1; \frac{3}{2}; x^2\right)$$

both of which combine in the single formula

$$p_n(x) = x^n F\left(-n, -\tfrac{1}{2}n + \tfrac{1}{2}; 1 - \beta - n; \frac{1}{x^2}\right)$$

which is the monic ultraspherical polynomial $\hat{P}_n^{(\beta)}(x)$.

Thus we get the special Hahn cases, i.e., the symmetric OPS whose derivative is also an OPS is either the Hermite or the ultraspherical polynomial set.

More generally if we replace α by $q^{\beta+1}$, β by $q^{\alpha+\beta+2}$, $e = a = 1$ and then let $q \to 1$ we see that the monic OPS of (3.7) becomes the symmetric OPS $\{\hat{S}_n(x)\}$ where

$$\hat{S}_{2n}^{(\alpha,\beta)}(x) = \frac{n!}{(\alpha + \beta + n + 1)_n} P_n^{(\alpha,\beta)}(2x^2 - 1)$$

$$\hat{S}_{2n+1}^{(\alpha,\beta)}(x) = \frac{n!}{(\alpha + \beta + n + 2)_n} xP_n^{(\alpha,\beta+1)}(2x^2 - 1).$$

It is known that the system $\{\hat{S}_n(x)\}$ is orthogonal on $(-1, 1)$ with respect to the weight function

$$|x|^{2\beta+1}(1 - x^2)^\alpha.$$

The case $\alpha = 0$ is due to Szegö [7].

In this case the operator

$$L : x^n \to \mu_n x^{n-1}$$

where

$$\mu_{2n} = n \frac{(\beta + 2)_{n-1}}{(\delta + 2)_n}, \quad \mu_{2n+1} = \frac{(\beta + 2)_n}{(\delta + 2)_n}$$

so that

$$L\hat{S}_{2n+2}^{(\alpha,\beta)}(x) = (n+1)\frac{(\beta+2)_n}{(\delta+2)_{n+1}}\hat{S}_{2n-1}^{(\alpha+\beta-\delta,\delta+2)}(x).$$

Similarly

$$L\hat{S}_{2n+1}^{(\alpha,\beta)}(x) = \frac{(\beta+2)_n}{(\delta+2)_n}\hat{S}_{2n}^{(\alpha+\beta-\delta,\delta+1)}(x).$$

Another interesting special case is the case $\beta = 0$, $\alpha = q^{\mu+1/2}$, $\delta = q^{\nu-1/2}$, $a = 1/1-q$ and then let $q \to 1$ we get that the operator

$$Lx^n = \mu_n x^{n-1}$$

where

$$\mu_{2n+1} = \frac{\left(\mu+\frac{3}{2}\right)_n}{\left(\nu+\frac{1}{2}\right)_n}, \quad \mu_{2n} = n\frac{\left(\mu+\frac{3}{2}\right)_{n-1}}{\left(\nu+\frac{1}{2}\right)_n}$$

takes the polynomial set $\{\hat{p}_n(x)\}$ where

$$\hat{p}_{2n}(x) = (-1)^n n!\, L_n^{(\mu-1/2)}(x^2)$$

$$\hat{p}_{2n+1}(x) = (-1)^n n!\, x L_n^{(\mu+1/2)}(x^2)$$

into the same set with μ replaced by ν. This polynomial set is the generalized Hermite polynomials due to Szego [6, p. 371] orthogonal on $(-\infty, \infty)$ with respect to the weight function $|x|^{2\mu} e^{-x^2}$.

Finally we note that the operator of type (1.1) of this note with $\mu_n = (1-q^n)\, b^{n-1}[\alpha]_n/[\gamma]_n$ take the non-symmetric OPS

$$J_n(q, \alpha, \beta; x) \to \frac{(1-q^n)[\alpha]_n}{[\gamma]_n} J_{n-1}(q, \gamma q, \beta q; bx).$$

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