ATTI ACCADEMIA NAZIONALE DEI LINCEI

CLASSE SCIENZE FISICHE MATEMATICHE NATURALI

RENDICONTI

H. S. GOPALAKRISHNA, V. S. SHETIYA

On the real part of the derivatives of certain analytic functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **59** (1975), n.1-2, p. 22–25.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_59_1-2_22_0>

Funzioni analitiche. — *On the real part of the derivatives of certain analytic functions.* Nota [(*)] di H. S. GOPALAKRISHNA [(**)] e V. S. SHETIYA, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia una funzione analitica nel disco unitario $E = \{z : |z| < 1\}$, $f(0) = 0$, $f'(0) = 1$. Sia $F(z) = \lambda z f'(z) + (1 - \lambda) f(z)$ dove $\lambda \in [0, 1]$. Se $\alpha, \beta \in [0, 1)$ e $\mathrm{Re} f'(z) > \alpha$ per $z \in E$, allora il raggio del disco nel quale $\mathrm{Re}\, F'(z) > \beta$ si determina generalizzando un precedente risultato di S. M. Bajpai–R. S. L. Srivastava e R. J. Libera–A. E. Livingston.

1. INTRODUCTION

Let S denote the class of functions $f(z)$ which are analytic in the unit disk $E = \{z : |z| < 1\}$ and are normalized by the conditions $f(0) = 0$, $f'(0) = 1$. For $\alpha \in [0, 1)$, let $\mathcal{P}(\alpha)$ denote the class of functions $P(z)$ analytic in E which satisfy $P(0) = 1$ and $\mathrm{Re}\, P(z) > \alpha$ for $z \in E$. $\mathcal{P}(0)$ is written simply as $\mathcal{P}$. If $f(z) \in S$ and $F(z) = \lambda z f'(z) + (1 - \lambda) f(z)$ for $z \in E$, where $\lambda \in [0, 1]$, we obtain, in this paper, a sharp estimate for the radius of the disk in which $\mathrm{Re}\, F'(z) > \beta$ whenever $f'(z) \in \mathcal{P}(\alpha)$ and $\alpha, \beta \in [0, 1)$. Our result generalizes earlier results of Bajpai and Srivastava [1] and Libera and Livingston [2].

2. We need the following Lemmas which are generalizations of the Lemmas proved by Libera and Livingston [2, Lemmas 1 and 2].

LEMMA 1. *Let $\lambda \in [0,1]$ and θ be real. If*

$$Q(z) = \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} + \frac{2\lambda e^{i\theta} z}{(1 - e^{i\theta} z)^2}, \qquad \textit{then, for} \quad |z| = r < 1,$$

$$\begin{aligned} \mathrm{Re}\, Q(z) &\geq I_1(r, \lambda) \qquad \textit{if} \quad 0 \leq r < r_0, \\ &\geq I_2(r, \lambda) \qquad \textit{if} \quad r_0 \leq r < 1, \end{aligned}$$

where

$$I_1(r, \lambda) = \frac{1 - 2\lambda r - r^2}{(1 + r)^2}, \qquad I_2(r, \lambda) = -\frac{[(1 - \lambda) - (1 + \lambda) r^2]^2}{4\lambda (1 - r^2)^2}$$

and

$$\begin{aligned} r_0 &= \frac{\sqrt{1 + 3\lambda^2} - 2\lambda}{1 - \lambda} \qquad \textit{if} \quad \lambda < 1, \\ &= 1/2 \qquad \textit{if} \quad \lambda = 1. \end{aligned}$$

(*) Pervenuta all'Accademia l'8 agosto 1975.
(**) Department of Mathematics, Karnatak University, Dharwar-580003 (India).

Proof. Let $z = re^{i\varphi}$. Then simplification yields

$$\operatorname{Re} Q(z) = \frac{[1 - 4\lambda r^2 - r^4] + 2r\,[(1+\lambda)\,r^2 - (1-\lambda)]\cos(\theta+\varphi)}{[1 - 2r\cos(\theta+\varphi) + r^2]^2}$$

$$= H(\varphi), \qquad \text{say.}$$

For fixed r, θ and λ, $H'(\varphi) = 0$ and so $H(\varphi)$ attains its minimum value only when φ is $-\theta$ or $\pi - \theta$ or φ_1, where

$$\cos(\theta + \varphi_1) = \frac{(1+\lambda) - 6\lambda r^2 - (1-\lambda)\,r^4}{2r\,[(1-\lambda) - (1+\lambda)\,r^2]} = I_3(r, \lambda), \qquad \text{say.}$$

But φ_1 exists only when $r \geq r_0$ since, otherwise, $| I_3(r, \lambda) | > 1$.

For $\quad r \in [0, 1)$, $H(\pi - \theta) - H(-\theta) = - \dfrac{4r\,[1 - r^2 + \lambda(1 + r^2)]}{(1 - r^2)^2} \leq 0.$

Thus, for $0 \leq r < r_0$, the minimum value of $H(\varphi)$ is $H(\pi - \theta)$ which is $I_1(r, \lambda)$.

On the other hand, for $r \geq r_0$,

$$H(\varphi_1) - H(\pi - \theta) = - \frac{[(1-\lambda)\,r^2 + 4\lambda r - (1+\lambda]^2}{4\lambda\,(1 - r^2)^2} \leq 0$$

and so the minimum value of $H(\varphi)$ is $H(\varphi_1)$ which is $I_2(r, \lambda)$. This proves the lemma.

LEMMA 2. *If* $P(z) \in \mathcal{P}$ *and* $\lambda \in [0,1]$, *then, for* $|z| = r < 1$,

$$\operatorname{Re}\{\lambda z P'(z) + P(z)\} \geq I_1(r, \lambda) \quad \text{if} \quad 0 \leq r < r_0,$$

$$\geq I_2(r, \lambda) \quad \text{if} \quad r_0 \leq r < 1.$$

These inequalities are sharp for each λ.

Proof. By the well-known Herglotz-Stieltjes representation [3] we have

$$P(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{i\theta}}{1 - ze^{i\theta}}\, d\alpha(\theta),$$

where $\alpha(\theta)$ is real-valued and nondecreasing in $[0, 2\pi]$ with

$$\int_0^{2\pi} d\alpha(\theta) = 2\pi.$$

Thus,

$$\operatorname{Re}\{\lambda z P'(z) + P(z)\} = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} Q(z)\, d\alpha(\theta),$$

from which the lemma follows as an easy consequence of Lemma 1.

Choosing $P(z) = \frac{1 - ze^{i\theta}}{1 + ze^{i\theta}}$ where $\theta = 0$ if $0 \leq r < r_0$ and $\theta = \cos^{-1} I_3(r, \lambda)$ if $r_0 \leq r < 1$, it is easily seen that the bounds given in the lemma are rendered sharp at $z = r$.

In particular, for $\lambda = 1/2$, Lemmas 1 and 2 reduce to Lemmas 1 and 2 respectively of Libera and Livingston [2].

THEOREM A. *Let* $\alpha, \beta \in [0, 1)$ *and* $\lambda \in [0,1]$.

For $r \in [0, 1)$, *let*

$$N(r) = (1 - \beta) + 2\{(\alpha - \beta) - \lambda(1 - \alpha)\} r + (2\alpha - \beta - 1) r^2, \tag{1}$$

$$\begin{aligned} M(r) = {} & \{2(1 + \alpha - 2\beta)\lambda - (1 - \alpha)(1 + \lambda^2)\} + \\ & + 2\{(1 - \alpha)(1 - \lambda^2) - 4(\alpha - \beta)\lambda\} r^2 + \\ & + \{2\lambda(3\alpha - 2\beta - 1) - (1 - \alpha)(1 + \lambda^2)\} r^4. \end{aligned} \tag{2}$$

Let, further, for $1 + \beta - 2\alpha > 0$,

$$\begin{aligned} L(\alpha, \beta, \lambda) = {} & (1 + \beta - 2\alpha) r_0 - \alpha(1 + \lambda) + \lambda + \beta - \\ & - [\{\alpha(1 + \lambda) - \lambda - \beta\}^2 + (1 + \beta - 2\alpha)(1 - \beta)]^{1/2}. \end{aligned} \tag{3}$$

Let $f(z) \in S$ *and* $F(z) = \lambda z f'(z) + (1 - \lambda) f(z)$ *for* $z \in E$. *If* $f'(z) \in \mathcal{P}(\alpha)$, *then* $\operatorname{Re} F'(z) > \beta$.

(i) *for* $|z| < r_1$ *where* r_1 *is the smallest positive root of the equation* $N(r) = 0$ *when* $2\alpha - \beta - 1 < 0$ *and* $L(\alpha, \beta, \lambda) \geq 0$, *and*

(ii) *for* $|z| < r_2$ *where* r_2 *is the smallest root greater than* r_0 *of the equation* $M(r) = 0$ *when* $2\alpha - \beta - 1 \geq 0$ *or* $2\alpha - \beta - 1 < 0$ *and* $L(\alpha, \beta, \lambda) < 0$, r_0 *being as defined in Lemma* 1.

The above results are sharp.

Proof. Since $f'(z) \in \mathcal{P}(\alpha)$, there exists a function $P(z)$ in $\mathcal{P}$ such that $f'(z) = (1 - \alpha) P(z) + \alpha$. Then,

$$F'(z) = \lambda z f''(z) + f'(z) = (1 - \alpha)\{\lambda z P'(z) + P(z)\} + \alpha.$$

Hence

$$\operatorname{Re}\{F'(z) - \beta\} = (1 - \alpha) \operatorname{Re}\{\lambda z P'(z) + P(z)\} + (\alpha - \beta)$$

By Lemma 2, it follows that, for $|z| = r \leq r_0$,

$$\operatorname{Re}\{F'(z) - \beta\} \geq (1 - \alpha) I_1(r, \lambda) + \alpha - \beta = \frac{N(r)}{(1 + r)^2}, \tag{4}$$

and for $r_0 < r < 1$,

$$\operatorname{Re}\{F'(z) - \beta\} \geq (1 - \alpha) I_2(r, \lambda) + \alpha - \beta = \frac{M(r)}{4\lambda(1 - r^2)^2}. \tag{5}$$

It is easily verified that

$$I_1(r_0, \lambda) = I_2(r_0, \lambda) = -\lambda \left(\frac{1-r_0}{1+r_0}\right)^2,$$

so that $M(r_0) = N(r_0)$.

(i) Suppose, first, that $2\alpha - \beta - 1 < 0$. Then the equation $N(r) = 0$ has the unique positive root

$$r_1 = r_0 - \frac{L(\alpha, \beta, \lambda)}{1 + \beta - 2\alpha}.$$

Since $N(0) = 1 - \beta > 0$, it follows that $N(r) > 0$ for $r \in [0, r_1)$. Clearly $r_1 \leq r_0$ if and only if $L(\alpha, \beta, \lambda) \geq 0$. Hence it follows from (4) that $\operatorname{Re} F'(z) > \beta$ for $0 \leq r < r_1$ if $L(\alpha, \beta, \lambda) \geq 0$.

On the other hand, if $L(\alpha, \beta, \lambda) < 0$ then $r_1 > r_0$. So $M(r_0) = N(r_0) > 0$. Also, $M(1) = -4(1-\alpha)\lambda^2 \leq 0$. Therefore, $M(r)$ has a root in the interval $(r_0, 1]$ and if r_2 is the smallest root in this interval of $M(r)$, it follows that $M(r) > 0$ for $r_0 \leq r < r_2$. Hence, from (5), $\operatorname{Re} F'(z) > \beta$ for $r_0 \leq r < r_2$. Also, from (4), $\operatorname{Re} F'(z) > \beta$ for $r \in [0, r_0)$, since $r_1 > r_0$ and $N(r) > 0$ for $r \in [0, r_1)$. Thus $\operatorname{Re} F'(z) > \beta$ for $|z| < r_2$.

(ii) If $2\alpha - \beta - 1 \geq 0$, then $\alpha - \beta \geq 1 - \alpha \geq \lambda(1-\alpha)$ and so $N(r) \geq 1 - \beta > 0$ for all $r \geq 0$. From (4) and (5) it again follows that $\operatorname{Re} F'(z) > \beta$ for $|z| < r_2$.

That the above results are sharp follows from the fact that the bounds obtained in Lemma 2 are sharp so that equality can be achieved in (4) and (5). This completes the proof of Theorem A.

Remarks (i) For $\alpha = \beta$, $2\alpha - \beta - 1 = \alpha - 1 < 0$ and $L(\alpha, \beta, \lambda) \geq 0$. Hence it follows from Theorem A that if $f'(z) \in \mathscr{P}(\alpha)$ then $\operatorname{Re} F'(z) > \alpha$ for $|z| < \sqrt{\lambda^2 + 1} - \lambda$. For $\lambda = \frac{1}{1+c}$, where c is a positive integer, this reduces to a result of Bajpai and Srivastava [1, p. 159].

(ii) The particular case of Theorem A for $\lambda = 1/2$ has been considered by Libera and Livingston [2, Theorem 2].

For $\lambda = 0$ and $\alpha \leq \beta$ in Theorem A, we have $F(z) \equiv f(z)$, $2\alpha - \beta - 1 < 0$ and it is easily verified that $L(\alpha, \beta, \lambda) \geq 0$ and $r_1 = \frac{1-\beta}{1+\beta-2\alpha}$.

Thus we have the following

COROLLARY. *Let $f(z) \in S$ and $f'(z) \in \mathscr{P}(\alpha)$ where $0 \leq \alpha < 1$. If $\alpha \leq$ $\leq \beta < 1$, then $\operatorname{Re} f'(z) > \beta$ for $|z| < \frac{1-\beta}{1+\beta-2\alpha}$. The result is sharp.*

REFERENCES

[1] S. K. BAJPAI and R. S. L. SRIVASTAVA (1972) – *On the radius of convexity and starlikeness of univalent functions*, « Proc. Amer. Math. Soc. », *32*, 153–160.

[2] R. J. LIBERA and A. E. LIVINGSTON (1971) – *On the univalence of some classes of regular functions*, « Proc. Amer. Math. Soc. », *30*, 327–336.

[3] P. PORCELLI (1966) – *Linear Spaces of Analytic Functions*, Rand McNally, Chicago, Ill.