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Properties of θ-continuous functions

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Topologia. — *Properties of θ-continuous functions*. Nota di TAKASHI NOIRI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Vengono stabilite varie proprietà delle funzioni θ–continue, in relazione specialmente agli insiemi θ–chiusi di uno spazio topologico.

1. INTRODUCTION

The concept of θ-continuity was introduced and investigated by S. Fomin [1]. N. Veličko [4] defined θ-closed sets in a topological space and obtained several properties concerning H-closed spaces and θ-continuous functions. It is known that the concept of θ-continuity is stronger than that of weak-continuity [3] and is weaker than that of almost-continuity in the sense of Singal [2]. The purpose of the present paper is to obtain several properties of θ-continuous functions and to investigate some relations between θ-closed sets and θ-continuous functions.

2. DEFINITIONS

Let X be a topological space and S a subset of X. We shall denote the closure of S and the interior of S by Cl (S) and Int (S) respectively. The following definitions are due to N. Veličko [4]. A point $x \in X$ is called the *θ-cluster* point of S if $S \cap Cl(U) \neq \emptyset$ for every open set U containing x. The set of all θ-cluster points of S is called the *θ-closure* of S and is denoted by [S]. A set S is said to be *θ-closed* if S = [S]. It is obvious that every θ-closed set is closed. If S is open, then [S] = Cl (S). In a regular space, closedness and θ-closedness are coincident. A filter base $\mathcal{F}$ is said to be *θ-convergent* to a point $x \in X$ if for any open set U containing x, there exists an $F \in \mathcal{F}$ such that $F \subset Cl(U)$. A function $f: X \to Y$ is said to be *θ-continuous* [1] if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(Cl(U)) \subset Cl(V)$.

3. θ-CONTINUOUS FUNCTIONS

THEOREM 1. *A function* $f: X \to Y$ *is θ-continuous if and only if for each point* $x \in X$ *and each filter base* $\mathcal{F}$ *in* X *θ-converging to* x, *the filter base* $f(\mathcal{F})$ *is θ-convergent to* $f(x)$.

(*) Nella seduta dell'11 giugno 1975.

Proof. Necessity. Let $x \in X$ and $\mathcal{F}$ be any filter base in X θ-converging to x. Then, for any open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$ because f is θ-continuous. Since $\mathcal{F}$ is θ-convergent to x, there exists an $F \in \mathcal{F}$ such that $F \subset \mathrm{Cl}(U)$. Thus, we have $f(F) \subset \mathrm{Cl}(V)$. This implies that $f(\mathcal{F})$ is θ-convergent to $f(x)$.

Sufficiency. Let $x \in X$ and V be any open set in Y containing $f(x)$. Put $\mathcal{F} = \{\mathrm{Cl}(U) \mid U \text{ is open in } X, x \in U\}$. Then $\mathcal{F}$ is a filter base θ-converging to x. Hence, there exists $\mathrm{Cl}(U) \in \mathcal{F}$ such that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is θ-continuous.

THEOREM 2. *Let* $f: X \to Y$ *be a function and* $g: X \to X \times Y$ *be the graph function of* f, *given by* $g(x) = (x, f(x))$ *for every point* $x \in X$. *Then* f *is* θ-*continuous if and only if* g *is* θ-*continuous.*

Proof. Necessity. Suppose f is θ-continuous. Let $x \in X$ and W be any open set in $X \times Y$ containing $g(x)$. Then there exist open sets $R \subset X$ and and $V \subset Y$ such that $g(x) = (x, f(x)) \in R \times V \subset W$. Since f is θ-continuous, there exists an open set U in X containing x such that $U \subset R$ and $f(\mathrm{Cl}(U)) \subset \subset \mathrm{Cl}(V)$. Thus we have $g(\mathrm{Cl}(U)) \subset \mathrm{Cl}(R) \times \mathrm{Cl}(V) \subset \mathrm{Cl}(W)$. This implies that g is θ-continuous.

Sufficiency. Suppose g is θ-continuous. Let $x \in X$ and V be any open set in Y containing $f(x)$. Then $X \times V$ is an open set in $X \times Y$ containing $g(x)$. Since g is θ-continuous, there exists an open set U in X containing x such that $g(\mathrm{Cl}(U)) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. It follows from the definition of g that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is θ-continuous.

4. URYSOHN SPACES AND θ-CONTINUOUS FUNCTIONS

By a *θ-continuous retraction* we mean a θ-continuous function $f: X \to A$ where A is a subset of X and $f \mid A$ is the identity mapping on A. A space X is called a *Urysohn* space if for every pair of distinct points x and y in X, there exist open sets U and V in X such that $x \in U$, $y \in V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \varnothing$.

THEOREM 3. *Let* $A \subset X$ *and* $f: X \to A$ *be a* θ-*continuous retraction of* X *onto* A. *If* X *is Urysohn, then* A *is* θ-*closed in* X.

Proof. Suppose A is not θ-closed in X. Then there exists a point $x \in [A] - A$. Since f is a θ-continuous retraction, we have $f(x) \neq x$. Since X is Urysohn, there exist open sets U and V in X such that $x \in U$, $f(x) \in V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \varnothing$. Now, let W be any open set in X containing x. Then $U \cap W$ is an open set containing x and hence $A \cap \mathrm{Cl}(U \cap W) \neq \varnothing$ because $x \in [A]$. Therefore, there exists a point $y \in A \cap \mathrm{Cl}(U \cap W)$. Since $y \in A$, $f(y) = y \in \mathrm{Cl}(U)$ and hence $f(y) \notin \mathrm{Cl}(V)$. This shows that $f(\mathrm{Cl}(W)) \not\subset \mathrm{Cl}(V)$. This contradicts the hypothesis that f is θ-continuous. Thus A is θ-closed in X.

THEOREM 4. *If* Y *is a Urysohn space and* $f: X \rightarrow Y$ *is a θ-continuous injection, then* X *is Urysohn.*

Proof. Let x_1 and x_2 be any distinct points of X. Then, we have $f(x_1) \neq f(x_2)$ because f is injective. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. It follows from the θ-continuity of f that there exists an open set U_j in X containing x_j such that $f(\mathrm{Cl}(U_j)) \subset \mathrm{Cl}(V_j)$, where $j = 1, 2$. Therefore, we obtain $\mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) \subset f^{-1}(\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)) = \varnothing$. This shows that X is Urysohn.

THEOREM 5. *If* f_1 *and* f_2 *are θ-continuous functions of a space* X *into a Urysohn space* Y, *then the set* $\{x \in X \mid f_1(x) = f_2(x)\}$ *is θ-closed in* X.

Proof. By A we denote the set $\{x \in X \mid f_1(x) = f_2(x)\}$. If $x \in X - A$, then we have $f_1(x) \neq f_2(x)$. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f_1(x) \in V_1$, $f_2(x) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. Since f_j is θ-continuous, there exists an open set U_j in X containing x such that $f_j(\mathrm{Cl}(U_j)) \subset \mathrm{Cl}(V_j)$, where $j = 1, 2$. Put $U = U_1 \cap U_2$, then U is an open set in X containing x and $f_1(\mathrm{Cl}(U)) \cap f_2(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. This implies that $\mathrm{Cl}(U) \cap A = \varnothing$ and hence $x \notin [A]$. Therefore, we obtain $[A] \subset A$. This shows that A is θ-closed in X.

THEOREM 6. *If* Y *is a Urysohn space and* $f: X \rightarrow Y$ *is a θ-continuous function, then the set* $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ *is θ-closed in the product space* $X \times X$.

Proof. By A we denote the set $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$. If $(x_1, x_2) \in X \times X - A$, then we have $f(x_1) \neq f(x_2)$. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \varnothing$. Since f is θ-continuous, there exists an open set U_j containing x_j such that $f(\mathrm{Cl}(U_j)) \subset \mathrm{Cl}(V_j)$, where $j = 1, 2$. Put $U = U_1 \times U_2$, then U is an open set in $X \times X$ containing (x_1, x_2) and $A \cap \mathrm{Cl}(U) = \varnothing$. Therefore, we have $(x_1, x_2) \in X \times X - [A]$. Thus, we obtain $[A] \subset A$. This shows that A is θ-closed in $X \times X$.

For a function $f: X \rightarrow Y$, the subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and is denoted by $G(f)$. It is well known that if $f: X \rightarrow Y$ is continuous and Y is Hausdorff, then $G(f)$ is closed in $X \times Y$. The following theorem is a modification of this result.

LEMMA. *Let* $f: X \rightarrow Y$ *be a function. Then* $G(f)$ *is θ-closed in* $X \times Y$ *if and only if for each point* $(x, y) \in X \times Y - G(f)$, *there exist open sets* U *and* V *containing* x *and* y *respectively such that* $f(\mathrm{Cl}(U)) \cap \mathrm{Cl}(V) = \varnothing$.

Proof. This follows easily from that for any open sets U and V containing x and y respectively, $\mathrm{Cl}(U \times V) \cap G(f) = \varnothing$ if and only if $f(\mathrm{Cl}(U)) \cap \mathrm{Cl}(V) = \varnothing$.

THEOREM 7. *If* Y *is a Urysohn space and* $f: X \to Y$ *is a* θ*-continuous function, then* $G(f)$ *is* θ*-closed in* $X \times Y$.

Proof. (x, y) be any point of $X \times Y - G(f)$, then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W such that $y \in V$, $f(x) \in W$ and $Cl(V) \cap Cl(W) = \varnothing$. Since f is θ-continuous, there exists an open set U in X containing x such that $f(Cl(U)) \subset Cl(W)$. Therefore, we obtain $f(Cl(U)) \cap Cl(V) = \varnothing$. By Lemma, $G(f)$ is θ-closed in $X \times Y$.

THEOREM 8. *Let* $f: X \to Y$ *be a function with the* θ*-closed graph. If* f *is surjective (resp. injective), then* Y *(resp.* X*) is Hausdorff.*

Proof. Suppose f is surjective. Let y and z be any distinct points of Y. Then there exists a point $x \in X$ such that $f(x) = z$, thus $(x, y) \notin G(f)$. Since $G(f)$ is θ-closed, by Lemma, there exist open sets U and V such that $x \in U$, $y \in V$ and $f(Cl(U)) \cap Cl(V) = \varnothing$. Therefore $Y - Cl(V)$ is an open set containing z. This implies that Y is Hausdorff. Next, suppose f is injective. Let x and w be any distinct points of X, then $f(x) \neq f(w)$ and hence $(x, f(w)) \notin G(f)$. Therefore, by Lemma, there exist open sets U and V containing x and $f(w)$ respectively such that $f(Cl(U)) \cap Cl(V) = \varnothing$. Thus $X - Cl(U)$ is an open set of X containing w. This implies that X is Hausdorff.

5. PRODUCT SPACES AND θ-CONTINUOUS FUNCTIONS

Let $\{X_\alpha \mid \alpha \in \mathscr{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathscr{A}\}$ be any families of spaces with the same set $\mathscr{A}$ of indices. We shall simply denote the product spaces $\Pi\{X_\alpha \mid \alpha \in \mathscr{A}\}$ and $\Pi\{Y_\alpha \mid \alpha \in \mathscr{A}\}$ by ΠX_α and ΠY_α respectively.

THEOREM 9. *A function* $f: X \to \Pi Y_\alpha$ *is* θ*-continuous if and only if* $p_\beta \circ f: X \to Y_\beta$ *is* θ*-continuous for each* $\beta \in \mathscr{A}$*, where* p_β *is the projection of* ΠY_α *onto* Y_β.

Proof. Necessity. Suppose f is θ-continuous. Since p_β is continuous for each $\beta \in \mathscr{A}$ and the composition of θ-continuous functions is θ-continuous, $p_\beta \circ f$ is θ-continuous for each $\beta \in \mathscr{A}$.

Sufficiency. Suppose $p_\beta \circ f$ is θ-continuous for each $\beta \in \mathscr{A}$. Let $x \in X$ and V be any open set in ΠY_α containing $f(x)$. Then there exists a basic open set V_0 such that $f(x) \in V_0 \subset V$ and $V_0 = \prod_{j=1}^{n} V_{\alpha_j} \times \prod_{\beta \neq \alpha_j} Y_\beta$, where V_{α_j} is an open set of Y_{α_j} for each j $(1 \leq j \leq n)$. Since $p_\beta \circ f$ is θ-continuous for each $\beta \in \mathscr{A}$, for each j there exists an open set U_j in X containing x such that $(p_{\alpha_j} \circ f)(Cl(U_j)) \subset Cl(V_{\alpha_j})$. Put $U = \bigcap_{j=1}^{n} U_j$, then U is an open set in X containing x and

$$f(\mathrm{Cl}(\mathrm{U})) \subset f\left(\bigcap_{j=1}^{n} \mathrm{Cl}(\mathrm{U}_j)\right) \subset \bigcap_{j=1}^{n} p_{\alpha_j}^{-1}(p_{\alpha_j}\circ f)(\mathrm{Cl}(\mathrm{U}_j)) \subset$$

$$\subset \bigcap_{j=1}^{n} p_{\alpha_j}^{-1}(\mathrm{Cl}(\mathrm{V}_{\alpha_j})) = \mathrm{Cl}(\mathrm{V}_0).$$

This shows that f is θ-continuous.

THEOREM 10. *Let $f_\alpha : \mathrm{X}_\alpha \to \mathrm{Y}_\alpha$ be a function for each $\alpha \in \mathscr{A}$ and $f : \Pi\mathrm{X}_\alpha \to \Pi\mathrm{Y}_\alpha$ a function defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each point (x_α) in $\Pi\mathrm{X}_\alpha$. Then f is θ-continuous if and only if f_α is θ-continuous for each $\alpha \in \mathscr{A}$.*

Proof. Necessity. Suppose f is θ-continuous. Let α be any fixed element of $\mathscr{A}$. Let $x_\alpha \in \mathrm{X}_\alpha$ and V_α be any open set in Y_α containing $f_\alpha(x_\alpha)$. Then there exists a point $x \in \Pi\mathrm{X}_\beta$ such that $p_\alpha(x) = x_\alpha$, where p_α is the projection of $\Pi\mathrm{X}_\beta$ onto X_α. Since $\mathrm{V} = \mathrm{V}_\alpha \times \prod_{\beta \neq \alpha} \mathrm{Y}_\beta$ is an open set in $\Pi\mathrm{Y}_\beta$ containing $f(x)$, there exists a basic open set U in $\Pi\mathrm{X}_\beta$ containing x such that $f(\mathrm{Cl}(\mathrm{U})) \subset \mathrm{Cl}(\mathrm{V})$ because f is θ-continuous. Put $\mathrm{U}_\alpha = p_\alpha(\mathrm{U})$, then we obtain

$$f_\alpha(\mathrm{Cl}(\mathrm{U}_\alpha)) = f_\alpha(p_\alpha(\mathrm{Cl}(\mathrm{U}))) = (q_\alpha \circ f)(\mathrm{Cl}(\mathrm{U})) \subset q_\alpha(\mathrm{Cl}(\mathrm{V})) = \mathrm{Cl}(\mathrm{V}_\alpha),$$

where q_α is the projection of $\Pi\mathrm{Y}_\beta$ onto Y_α. This shows that f_α is θ-continuous.

Sufficiency. Suppose f_α is θ-continuous for each $\alpha \in \mathscr{A}$. Let $x = (x_\alpha) \in \Pi\mathrm{X}_\alpha$ and V be any open set in $\Pi\mathrm{Y}_\alpha$ containing $f(x)$. Then, there exists a basic open set V_0 such that $f(x) \in \mathrm{V}_0 \subset \mathrm{V}$ and $\mathrm{V}_0 = \prod_{j=1}^{n} \mathrm{V}_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} \mathrm{Y}_\alpha$, where V_{α_j} is open in Y_{α_j} for each j $(1 \leqq j \leqq n)$. Since f_α is θ-continuous for each $\alpha \in \mathscr{A}$, there exists an open set U_{α_j} in X_{α_j} containing x_{α_j} such that $f_{\alpha_j}(\mathrm{Cl}(\mathrm{U}_{\alpha_j})) \subset \subset \mathrm{Cl}(\mathrm{V}_{\alpha_j})$, where $j = 1, 2, \cdots, n$. Put $\mathrm{U} = \prod_{j=1}^{n} \mathrm{U}_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} \mathrm{X}_\alpha$, then U is an open set in $\Pi\mathrm{X}_\alpha$ containing x and $f(\mathrm{Cl}(\mathrm{U})) \subset \mathrm{Cl}(\mathrm{V}_0) \subset \mathrm{Cl}(\mathrm{V})$. This shows that f is θ-continuous.

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