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Some applications of Darbo's theorem

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Topologia. — *Some applications of Darbo's theorem.* Nota di KANHAYA LAL SINGH, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Usufruendo di un teorema di Darbo [2], vengono dimostrati due teoremi concernenti le contrazioni di k-insiemi. Più precisamente, il Teorema 2.1 stabilisce una proprietà di surgettività simile a quella del teorema di Browder [3], ed il Teorema 2.2 assicura l'esistenza di punti fissi per la somma di due applicazioni. Come corollari di quest'ultimo teorema si ottengono fra l'altro i risultati di Nashed e Wong [4], Sing [10], Rienermann [8], Edmund [5], Kachuraskii, Krasnoselskii e Zabreico [11].

The notion of measure of noncompactness was introduced by C. Kuratowski [1] as follows:

DEFINITION 1.1. Let X be a (real) Banach space. Let D be a bounded subset of X. Then the *measure of noncompactness* of D, denoted by $\gamma(\mathrm{D})$ is defined as

$\gamma(\mathrm{D}) = \inf\{\varepsilon > 0/\mathrm{D}$ can be covered by a finite number of subsets of diameter $< \varepsilon\}$.

$\gamma(\mathrm{D})$ has the following properties:

(1) $0 < \gamma(\mathrm{D}) < d(\mathrm{D})$, where $d(\mathrm{D})$ denotes the diameter of D,

(2) $\gamma(\mathrm{D}) = 0$ if and only if D is precompact,

(3) $\gamma(\mathrm{C} \cup \mathrm{D}) = \max\{\gamma(\mathrm{C}), \gamma(\mathrm{D})\}$,

(4) $\gamma(\mathrm{C}(\mathrm{D}, \varepsilon)) < \gamma(\mathrm{D}) + 2\varepsilon$, where $\mathrm{C}(\mathrm{D}, \varepsilon) = \{x \text{ in } \mathrm{X}/d(x, \mathrm{D}) < \varepsilon\}$,

(5) $\mathrm{C} \subset \mathrm{D}$ implies $\gamma(\mathrm{C}) < \gamma(\mathrm{D})$,

(6) $\gamma(\mathrm{C}+\mathrm{D}) < \gamma(\mathrm{C}) + \gamma(\mathrm{D})$, where $\mathrm{C}+\mathrm{D} = \{c + d/c \text{ in C and } d \text{ in D}\}$.

Closely related to the notion of measure of noncompactness is the concept of k-set contraction first defined by Darbo [2] as follows.

DEFINITION 2.1. Let X be a Banach space. Let D be a bounded subset of X. Let $\mathrm{T}: \mathrm{D} \to \mathrm{X}$ be a continuous mapping. T is said to be *k-set contraction* if $\gamma(\mathrm{T}(\mathrm{D})) < k\gamma(\mathrm{D})$ for some $k \geq 0$. If $k < 1$, i.e.

$$\gamma(\mathrm{T}(\mathrm{D})) < \gamma(\mathrm{D}),$$

T is called *densifying* (Furi and Vignoli [6]).

THEOREM A (Darbo). *Let* D *be a closed, bounded and convex subset of a Banach space* X. *Let* $\mathrm{T}: \mathrm{D} \to \mathrm{D}$ *be a k-set contraction with* $k < 1$. *Then* T *has a fixed point.*

(*) Nella seduta dell'11 giugno 1975.

THEOREM 1.1. *Let* X *be a reflexive Banach space and* X^* *be its dual space. Let* T *be a nonlinear operator (or not necessarily linear) that maps* X *into* X^*. *Suppose that* T *is strictly positive* ($(T(x), x) > 0$ *for all* x *in* X) *and a* k*-set contraction with* $k < 1$. *Then* T *is surjective.*

Proof. It is enough to show that $T(x) = x$ has a solution or equivalently $W(x) = x - T(x)$ has a fixed point. First we note that W is an α-set contraction with $\alpha < 1$. Indeed, let D be any bounded but not precompact subset of X, then by definition of W we have

$$W(D) = I(D) - T(D).$$

Hence

$$\gamma(W(D)) = \gamma(I(D) - T(D)) < \gamma(D) - k\gamma(D) =$$
$$= \alpha\gamma(D), \quad \text{where} \quad \alpha = 1 - k < 1.$$

Since T is strictly positive, therefore there exists an $r > 0$ such that $(T(x), x) > 0$ for all x in S_r, where $S_r = \{x \text{ in } X / \|x\| = r\}$. Now using the definition of W we have

$$(1) \qquad (W(x), x) = (x - T(x), x) = (x, x) - (T(x), x) <$$
$$< \|x\|^2 \text{ (since } (T(x), x) > 0).$$

Now define

$$F : X \to X^* \quad \text{as follows:} \quad F(x) = \begin{cases} W(x) & \text{if } \|W(x)\| < r \\ \dfrac{r\,W(x)}{\|W(x)\|} & \text{if } \|W(x)\| \geq r. \end{cases}$$

Then $F(x)$ is densifying. Inded, setting $f_1(x) = W(x)$, $f_2(x) = 0$, $\lambda_1(x) = \dfrac{r}{\|W(x)\|}$ for $\|W(x)\| \geq r$ and $\lambda_1(x) = 1$ for $\|W(x)\| < r$ and $\lambda_2(x) = 1 - \lambda_1(x)$ we have

$$F(x) = \lambda_1(x) f_1(x) + \lambda_2(x) f_2(x).$$

Hence by Theorem [9, Theorem 9, p. 17] $F(x)$ is α-set contraction with $\alpha < 1$. Moreover, clearly $F(B_r) \subset B_r$, where B_r is the ball of radius r around the origin. Thus by Darbo's Theorem [2] F has a fixed point x_0. Now we have two possibilities, either x_0 belongs to the interior of B_r or x_0 is on the boundary S_r.

Case 1. Suppose x_0 belongs to the interior of B_r. Then $F(x_0) = x_0 = W(x_0)$, i.e. x_0 is a fixed point of W as was claimed.

Case 2. Suppose x_0 belongs to the boundary of B_r, i.e. x_r lies on S_r. Then

$$F(x_0) = x_0 = \frac{r\,W(x_0)}{\|W(x_0)\|},$$

or

$$(x_0, x_0) = \frac{r\,(W(x_0), x_0)}{\|W(x_0)\|}.$$

Hence

$$\| W(x_0) \| \| x_0 \|^2 = r(W(x_0), x_0). \tag{2}$$

Using (1) we can write (2) as

$$\| W(x_0) \| \| x_0 \|^2 < r \| x_0 \|^2.$$

This implies $\| W(x_0) \| < r$, a contradiction to the fact that $\| W(x_0) \| \geq r$. Thus Theorem 2.1.

Remark 2.1. Theorem 2.1 remains true even if we assume T to be either densifying or 1-set contraction. But in both cases the auxiliary mapping W turns out to be 0-set contraction.

Remark 2.2. A theorem similar to 2.1 has been proved by Browder [3], where T is assumed to satisfy the condition of monotonicity emicontinuity and coerciveness.

Remark 2.3. A theorem similar to 2.1 for Hilbert space with the assumption that $I - T$ is coercive has been proved by Edmund and Webb [7]. At any event since every Hilbert space is reflexive, our theorem is more general than that of Edmund and Webb [7]. Moreover we do not require the coerciveness of $I - T$.

THEOREM 2.2. *Let* X *be a Banach space. Let* D *be a closed, bounded and convex subset of* X. *Let* $A, B : D \to X$ *be two mappings such that*

(1) A *is densifying,*

(2) B *is either weakly continuous or completely continuous. Then there exists a* x_0 *in* D *such that* $A(x_0) + B(x_0) = x_0$.

Proof. Without loss of generality we may assume that the origin zero belongs to D. Let k_n be a sequence of numbers such that $0 < k_n < 1$ for each n and $k_n \to 1$ as $n \to \infty$. Clearly $k_n A$ is a k_n-set contraction with $k_n < 1$. Since B is weakly continuous (completely continuous) and therefore B is a 0-set contraction. Thus we conclude that $T = k_n(A + B)$ is a k_n-set- with $k_n < 1$. Hence by Darbo's Theorem [2] for each n, there exists a point x_n in D such that $T(x_n) = k_n(A(x_n) + B(x_n)) = x_n$.

For the sequence $\{x_n\}$ thus determined we have

$$x_n - (A(x_n) + B(x_n)) = k_n(A(x_n) + B(x_n)) - (A(x_n) + B(x_n))$$
$$= (k_n - 1)[A(x_n) + B(x_n)] \to 0 \quad \text{as} \quad n \to \infty.$$

Since $k_n \to 1$ and $\{T(x_n)\} \subset D$ is bounded. Hence zero lies in the closure of $(I - T)(D)$. But since $I - T$ is closed (9, Lemma 1, pp. 80), therefore T has a fixed point in D i.e. $A + B$ has a fixed point D. Thus there exists some x_0 in D such that $A(x_0) + B(x_0) = x_0$.

Remark 2.4. If in Theorem 2.2 instead of assuming A to be densifying one assumes A to be 1-set contraction, then the assumption $(I - T)$ D is closed is enough to guarantee the existence of a point x_0 such that $A(x_0) + B(x_0) = x_0$.

DEFINITION 2.2. Let X be a Banach space. A mapping $T : X \to X$ is said to be *demiclosed* if for any sequence x_n such that $x_n \to x$ (i.e. x_n converges weakly to x) and $T(x_n) \to y$. Then $T(x) = y$.

LEMMA 2.1. *Let* X *be a uniformly convex Banach space. Let* D *be a closed, bounded and convex subset of* X. *Let* $T : D \to X$ *be nonexpansive mapping. Then the set* $(I - T) D$ *is closed.*

Proof. By the Theorem [14 Theorem, pp. 660] it follows that $(I - T)$ is demiclosed. To show that $(I - T) D$ is closed, let x_n be a sequence in D such that $(I - T)(x_n) \to x_0$. We need to show that x_0 lies in $(I - T) D$. Since X is uniformly convex, therefore it is reflexive. Now D being closed, bounded and convex is weakly compact. Since X is reflexive we can replace $\{x_n\}$ by some subsequence, which for bravity we denote by $\{x_n\}$ such that $x_n \to y_0$ for some y_0 in X. But D is weakly compact, therefore y_0 belongs to D. Hence by demiclosedness of $(I - T)$ we infer that $(I - T) x_0 = y_0$.

COROLLARY 2.1 ([8] Rienermann). *Let* X *be a uniformly convex Banach space and let* D *be a nonempty, closed, bounded and convex subset of* X. *Let*

$$f : D \to D , \quad g : D \to D , \quad h : D \to D ,$$

be such that

(a) $f = g + h$,

(b) $\| g(x) - g(y) \| \leq \| x - y \|$ *for all* x, y *in* D (i.e. g *is nonexpansive*),

(c) h *is strongly continuous, i.e. if* x_n *converges weakly to* x *then* $h(x_n)$ *converges strongly to* $h(x)$. *Then* $f = g + h$ *has at least one fixed point.*

Proof. Since g is nonexpansive, therefore it is 1-set contraction moreover $h(x)$ being strongly continuous is a 0-set contraction. Thus $f = g + h$ is 1-set contraction. Indeed, let A be any bounded but not precompact subset of D, then by definition of $f(x)$ we have

$$f(A) = g(A) + h(A) .$$

Therefore

$$\gamma f(A) = \gamma [g(A) + h(A)] \leq (A) .$$

Furthermore g being nonexpansive implies that $(I - T)$ is demiclosed, therefore by Lemma 2.1 we conclude that $(I - T) D$ is closed. Thus all the assumptions of Remark 2.4 are satisfied, hence the Corollary 2.1 follows from Remark 2.4.

COROLLARY 2.2 ([11], Kachuraskii, Krasnoselskii and Zabrieko). *Let* H *be a Hilbert space. Let* D *be a closed, bounded and convex subset of* H. *Let* $T : D \to D$ *be a nonlinear operator such that* $T = A + B$, *where* A *is nonexpansive and* B *is completly continuous. Then* T *has at least one fixed point in* D.

Proof. The Corollary 2.2 follows from Corollary 2.1 by using the fact that every Hilbert space is uniformly convex. Moreover in a Hilbert space if A is nonexpansive, then $(I - A)$ is demiclosed ([12], the proof of this fact may be found using monotonicity, without motononicity the proof is given in Opial [13]).

COROLLARY 2.3 ([5], Edmund). *Let* H *be a Hilbert space. Let* D *be closed, bounded and convex subset of* H. *Let* $T : D \to D$ *be a nonlinear operator such that* $T = A + B$, *where*

(1) $A(x) + B(y)$ *in* D *for all* x, y *in* D,

(2) A *is nonexpansive, and*

(3) B *is completely continuous.*

Then T *has a fixed point.*

Remark 2.5. In the proof of Lemma 2.1 infact uniformconvexity was just used to guarantee the fact that $(I - T)$ was demiclosed and the rest of the proof was based on the property of reflexivity. Thus if X is reflexive and $(I - T)$ is demiclosed, then for any bounded, closed and convex subset D of X, $(I - T) D$ is closed. Thus we have the following Corollary.

COROLLARY 2.4 ([10], Singh). *Let* X *be a reflexive Banach space and* A *and* B *be two mappings of* D *into* X, *where* D *is a nonempty, closed bounded and convex subset of* X *such that*

(1) A *is nonexpansive and* $(I - A)$ *is demiclosed, and*

(2) B *is completely continuous.*

Then there exists some x *in* D *such that* $A(x) + B(x) = x$.

COROLLARY 2.5 ([10], Singh). *Let* X *be a reflexive Banach space and let* A *and* B *two mappings of* D *into* X, *where* D *is nonempty, closed bounded and convex subset of* X. *If* A *is* 1-*set contraction and* $(I - A)$ *is demiclosed and* B *is completely continuous, then* $T = A + B$ *has a fixed point in* D.

DEFINITION 2.3. Let X a Banach space. Let D be a bounded, closed and convex subset of X. A mapping $T : D \to D$ is said to be a *nonlinear contraction* if

$$\| T(x) - T(y) \| \leq \varphi \| x - y \| \qquad \text{for all } x, y \text{ in } D,$$

where $\varphi(r)$ for $r \geq 0$ is monotone nondecreasing function with continuous on the right such that $\varphi(r) > r$ for all $r > 0$.

COROLLARY 2.6 ([4], Nashed and Wong). *Let* X *be a Banach space, Let* D *be a bounded, closed and convex subset of* X. *Let* A *and* B *be two operators on* D *into* X *such that* $A(x) + B(y)$ *in* D *for every pair of* x, y *in* D. *If* A *is nonlinear contraction and* B *is completely continuous, then the equation* $A(x) + B(x) = x$ *has a solution in* D.

Proof. We first note that A is densifying. Indeed, let C be a bounded but not precompact subset of D, such that $\gamma(C) > 0$, let us take $\varepsilon > \gamma(C)$. Then there exists a finite covering $\{C_1, C_2, C_3, \cdots, C_n\}$ of C such that $d(C_k) < \varepsilon$ (for $k = 1, 2, 3, \cdots, n$). Clearly

$$A(C) = \bigcup_{k=1}^{n} A(C_k).$$

Let $1 \leq k \leq n$ be fixed. Let x, y in C_k, then clearly $\|x - y\| < \varepsilon$. Hence $\|A(x) - A(y)\| \leq \varphi \|x - y\| < \varphi(\varepsilon)$. Therefore $d(A(C_k)) \leq \varphi(\varepsilon)$. Thus $\gamma(A(C)) \leq \varphi(\varepsilon)$. If $\varepsilon \downarrow \gamma(A)$, then by the right continuity of φ we have

$$\gamma(A(C)) \leq \varphi(\gamma(A)) < \gamma(A).$$

Now B being completely continuous is 0-set contraction, therefore $A + B$ is densifying. Thus the result follows from Theorem 2.2.

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