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Abstract integral equations of Volterra type

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Equazioni funzionali. — *Abstract integral equations of Volterra type.* Nota di SERGIU AIZICOVICI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Siano H uno spazio di Hilbert reale, $a(t)$ una funzione reale su $[0, +\infty[$ ed $A(t)$ $(t \geq 0)$ una famiglia di operatori non lineari a più valori su H. In questo lavoro si studia l'equazione integrale di Volterra:

$$u(t) + \int_0^t A(s)\, u(s)\, ds \ni f(t) \qquad (0 \leq t < +\infty),$$

dove $f: [0, +\infty[\to H$ è una funzione assegnata. Fra l'altro, si ottengono teoremi di esistenza che generalizzano risultati di Barbu [1] coll'impiego di metodi di monotonicità.

1. INTRODUCTION

This paper is concerned with the existence and behaviour of solutions of the integral equation

$$u(t) + \int_0^t a(t-s)\, A(s)\, u(s)\, ds \ni f(t), \qquad 0 \leq t < +\infty. \tag{1.1}$$

Here u, f are functions with values in a real Hilbert space H, $a(t)$ is a scalar kernel and $A(t)$, for each $t \geq 0$, belongs to a class of maximal monotone graphs in $H \times H$.

Eq (1.1) has been thoroughly studied in the literature in the case where $H = R$ and $A(t)$ do not depend on t (see e.g. [4], [5]). MacCamy and Wong [6] investigated equations of a related form pointing out the role played by positive functions as convolution kernels of the Volterra operators. In [3], Friedman and Shinbrot have proved existence, uniqueness and differentiability theorems for solutions of (1.1) under the assumption that $A(t)$ are linear, unbounded operators in a Banach space X, with domain independent of t.

In a recent paper [1], Barbu has presented a discussion of the abstract equation

$$u(t) + \int_0^t a(t-s)\, Au(s)\, ds \ni f(t), \qquad t \geq 0, \tag{1.2}$$

where $A = \partial\varphi$ is the subdifferential of a convex, lower semicontinuous function $\varphi: H \to]-\infty, +\infty]$. His idea was to approach Eq. (1.2) through the theory of monotone operators.

Our purpose is to extend the method of [1] to the time dependent case.

(*) Nella seduta dell'11 giugno 1975.

2. PRELIMINARIES AND NOTATION

Throughout this paper H will denote a real Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$.

Consider a nonlinear multivalued operator (graph) A from H into itself, that is a subset of $H \times H$. We will use the notations

$$D(A) = \{x \in H ; Ax \neq \varphi\} \quad , \quad R(A) = \cup \{Ax ; x \in D(A)\} \quad , \quad A^{-1} y = \{x \in H ; y \in Ax\}. \tag{2.1}$$

DEFINITION 2.1. The operator A is said to be monotone on H provided that

$$\langle y_1 - y_2 , x_1 - x_2 \rangle \geq 0 , \qquad \text{for all} \quad x_i \in D(A) , y_i \in Ax_i , \qquad i = 1 , 2 . \tag{2.2}$$

If in addition it admits no proper monotone extensions, we say that A is maximal monotone.

If (2.2) is strengthened to

$$\langle y_1 - y_2 , x_1 - x_2 \rangle > 0 , \qquad \text{for all} \quad y_i \in Ax_i , i = 1 , 2 , \qquad \text{with} \quad x_1 \neq x_2 , \tag{2.3}$$

then A is called strongly monotone.

DEFINITION 2.2. Let $\varphi : H \to]-\infty, +\infty]$ be a lower semicontinuous, convex function, $\varphi \not\equiv +\infty$. We set

$$\operatorname{Dom} \varphi = \{x \in H ; \varphi(x) < +\infty\} . \tag{2.4}$$

The subdifferential $\partial\varphi$ of φ is defined by

$$\partial\varphi(x) = \{y \in H ; \varphi(u) - \varphi(x) \geq \langle y , u - x \rangle , \forall u \in H\} . \tag{2.5}$$

It is well known (see e.g. [2]) that $\partial\varphi$ is maximal monotone in $H \times H$.

In the following lemmas we collect for later use some elementary properties of maximal monotone operators.

LEMMA 2.1. *(i) A monotone operator* $A : D(A) \subset H \to 2^H$ *is maximal monotone if and only if* $R(I + \lambda A) = H$ *for any* $\lambda > 0$.

Here, as is usual I *stands for the identity operator on* H.

(ii) For each $\lambda > 0$ *define*

$$J_\lambda = (I + \lambda A)^{-1} \quad , \quad A_\lambda = \frac{1}{\lambda}(I - J_\lambda) . \tag{2.6}$$

Then J_λ *is a monotone contraction on* H, *while* A_λ *is maximal monotone, Lipschitz continuous with Lipschitz constant* $1/\lambda$.

LEMMA 2.2. *Let* φ *be a lower semicontinuous, convex mapping from* H *into* $]-\infty, +\infty]$, *nonidentically* $+\infty$. *Then the function*

$$\varphi_\lambda(u) = \inf_{v \in H} \left\{ \frac{\| u - v \|^2}{2\lambda} + \varphi(v) \right\}, \qquad u \in H ; \qquad \lambda > 0 , \tag{2.7}$$

is convex, Fréchet differentiable and $\partial\varphi_\lambda = (\partial\varphi)_\lambda$. *Moreover,*

$$\varphi_\lambda(u) = \varphi(J_\lambda u) + \frac{\lambda}{2} \| A_\lambda u \|^2, \qquad u \in H , \tag{2.8}$$

$$\varphi(J_\lambda u) \leq \varphi_\lambda(u) \leq \varphi(u) , \qquad u \in H . \tag{2.9}$$

The proofs of Lemmas 2.1, 2.2 may be found in [2]. We close this section with other usual notations and definitions.

DEFINITION 2.3. If $T \in]0, +\infty]$, $L^2(0, T; H)$ is the space of all (classes of) strongly measurable functions $u :]0, T[\to H$ such that

$$\| u \|^2_{L^2(0,T:H)} = \int_0^T \| u(t) \|^2 \, dt < +\infty. \tag{2.10}$$

For $T = +\infty$, denote by $L^2_{loc}(0, \infty : H)$ the corresponding local space, i.e.,

$$L^2_{loc}(0, \infty : H) = \{ u ; u \in L^2(0, T : H) \quad \text{for every} \quad 0 < T < +\infty \}. \tag{2.11}$$

DEFINITION 2.4. For each $T, 0 < T \leq +\infty$ we use the notation $W^{1,2}(0, T : H)$ to indicate the space of H-valued distributions u on $]0, T[$ satisfying

$$u, u' \in L^2(0, T; H). \tag{2.12}$$

Here u' denotes the distributional derivative of u.

Finally, set

$$W^{1,2}_{loc}(0, \infty; H) = \{ u ; u \in W^{1,2}(0, T; H), \forall T \in]0, +\infty[\}. \tag{2.13}$$

We recall that every $u \in W^{1,2}_{loc}(0, \infty; H)$ is locally absolutely continuous on $[0, +\infty[$, almost everywhere differentiable and its ordinary derivative coincides with u', for almost all $t > 0$.

3. STATEMENT OF RESULTS

Let $a : [0, +\infty[\to R$ and $\varphi : [0, +\infty[\times H \to]-\infty, +\infty]$ be subject to the following conditions:

(I) $a(t)$ is continuous on $[0, +\infty[$ and continuously differentiable on $]0, +\infty[$ satisfying

$$(-1)^k a^{(k)}(t) \geq 0, \qquad k = 0, 1, t > 0. \tag{3.1}$$

$$a'(t) \text{ is nondecreasing,} \tag{3.2}$$

$$a(t) \not\equiv 0, \quad t \geq 0. \tag{3.3}$$

(II) For each $t \geq 0$, the function $u \to \varphi(t, u)$ is convex, lower semicontinuous and nonidentically $+\infty$. Moreover, there exist two functions: $k : H \to [0, +\infty[$, Lipschitzian with Lipschitz constant equal to ρ and

$$b \in W^{1,2}_{loc}(0, \infty; R),$$

such that

$$\varphi^*(t, u) \leq \varphi^*(s, u) + k(u) | b(t) - b(s) |, \qquad \text{for all} \quad s, t \geq 0. \tag{3.4}$$

Here $\varphi^*(t, \cdot)$ denotes the conjugate function of $\varphi(t, \cdot)$, namely

$$\varphi^*(t, v) = \sup_{u \in H} \{ \langle u, v \rangle - \varphi(t, u) \}. \tag{3.5}$$

Remark 3.1. Hypotheses (I) imply (see [6]) that $a(t)$ defines a positive kernel, i.e.,

$$\int_0^T \left\langle u(t), \int_0^t a(t-s)\,u(s)\,ds \right\rangle dt \geq 0, \tag{3.6}$$

for all $u \in C(0, \infty; H)$ and $T \geq 0$, where $C(0, \infty; H)$ stands for the space of continuous functions on $[0, \infty[$ with values in H.

Remark 3.2. Conditions (II) were introduced by Peralba [7]. Obviously from (3.5) it follows

$$\operatorname{Dom} \varphi^*(t, \cdot) = D, \quad \text{independent of } t. \tag{3.7}$$

Consider next the equation

$$u(t) + \int_0^t a(t-s)\,A(s)\,u(s)\,ds \ni f(t), \qquad 0 \leq t < +\infty, \tag{3.8}$$

where $A(t)x = \partial\varphi(t, x)$, $t \geq 0$, $x \in H$.

DEFINITION 3.1. A function $u: [0, +\infty[\to H$ is called a solution to Eq. (3.8) if

$$u \in W^{1,2}_{loc}(0, \infty; H)\,;\; u(t) \in D(A(t)), \text{ a.e. on}]0, +\infty[, \tag{3.9}$$

and there exists $v \in L^2_{loc}(0, \infty; H)$ such that

$$v(t) \in A(t)\,u(t), \text{ a.e. on}]0, +\infty[, \tag{3.10}$$

$$u(t) + \int_0^t a(t-s)\,v(s)\,ds = f(t), \qquad t \geq 0. \tag{3.11}$$

For simplicity we will sometimes write $A(t)\,u(t)$ instead of $v(t)$.
We can now state our basic results:

THEOREM 1. *Suppose that (I) and (II) hold. Let $f: [0, +\infty[\to H$ satisfy*

$$f \in W^{1,2}_{loc}(0, \infty; H) \quad , \quad f(0) \in \operatorname{Dom} \varphi(0, \cdot). \tag{3.12}$$

Then Eq. (3.8) *has at least one solution $u(t)$ such that*

$$t \to \varphi(t, u(t)) \tag{3.13}$$

is absolutely continuous on every interval $[0, T]$.
If $A(t)$ is strictly monotone, a.e. on $]0, +\infty[$, *the solution $u(t)$ is unique.*

THEOREM 2. *Let (I), (II) and* (3.12) *be fulfilled. In addition suppose*

(3.14) $$a(\infty) = \lim_{t\to\infty} a(t) > 0,$$

(3.15) $$b' \in L^2(0, \infty; R),$$

(3.16) $$\varphi(t, u) \geq \omega(u) \quad \text{for all} \quad u \in H, t \geq 0,$$

where $\omega : H \to R$ *is such that* $\omega(u) \to +\infty$ *as* $\|u\| \to +\infty$.

(3.17) *There exist* $p_0 \in D$ *and a real constant* M *such that* $\varphi^*(t, p_0) \leq M$ *for all* $t \geq 0$.

(3.18) $$f' \in L^2(0, \infty; H).$$

Then Eq. (3.8) *has a solution satisfying*

(3.19) $$\|u(t)\| \quad \text{bounded on} \quad [0, +\infty[,$$

(3.20) $$u'(t), A(t)u(t) \in L^2(0, \infty; H)$$

(3.21) $$\varphi(t, u(t)) \text{ is absolutely continuous on } [0, +\infty[\text{ and } \lim_{t\to\infty} \varphi(t, u(t)) < +\infty.$$

THEOREM 3. *Besides (I), (II),* (3.12), (3.14), (3.15) *assume that* (3.17) *holds with* $p_0 = 0$ *and that*

(3.22) $$\int_0^\infty |a(t)|\, dt < +\infty,$$ (3.23) $$f \in W^{1,2}(0, \infty; H).$$

Then there exists at least one solution of Eq. (3.8) *satisfying* (3.21) *and*

(3.24) $$u \in W^{1,2}(0, \infty; H) \quad ; \quad A(t)u(t) \in L^2(0, \infty; H).$$

COROLLARY. *Suppose that Hypotheses (II) hold. Let* $a(t)$ *be a continuous scalar function on* $]0, +\infty[$ *such that*

(3.25) $$a \text{ is negative, nondecreasing on }]0, +\infty[,$$

(3.26) $$m + \int_0^t a(s)\, ds \geq 0, \qquad 0 < t < +\infty,$$

for some positive constant m.

If $f : [0, +\infty[\to H$ *and* $u_0 \in H$ *satisfy*

(3.27) $$f \in L^2_{loc}(0, \infty; H); \int_0^t f(s)\, ds \in L^2_{loc}(0, \infty; H) \qquad \text{for each} \quad t > 0,$$

(3.28) $$u_0 \in \text{Dom}\, \varphi(0, \cdot),$$

then there exists a solution (in the sense of Definition (3.1)) *of the integro-differential equation*

$$(3.29)\qquad u'(t) + m\,\partial\varphi(t, u(t)) + \int_0^t a(t-s)\,\partial\varphi(s, u(s))\,ds \ni f(t), \quad \text{a.e. on }]0, +\infty[$$

$$u(0) = u_0$$

such that

$$(3.20)\qquad t \to \varphi(t, u(t)) \text{ is absolutely continuous on each } [0, T].$$

If in addition (3.14)-(3.17) *are fulfilled and*

$$(3.31)\qquad m + \int_0^\infty a(s)\,ds > 0,$$

$$(3.32)\qquad f \in L^2(0, \infty; H),$$

then Eq. (3.29) *has a solution satisfying the conclusions of Theorem* 2.

4. PROOFS

In the proofs we shall need the following results due to Peralba [8].

LEMMA 4.1. *Let* φ: $[0, +\infty[\, x H \to]-\infty, +\infty]$ *satisfy Conditions* (*II*). *Suppose we are given* $T \in]0, \infty[$, $u: [0, T] \to H$ *and* $g:]0, T[\to H$ *such that*

$$(4.1)\qquad u \in W^{1,2}(0, T; H)$$

$$(4.2)\qquad g \in L^2(0, T; H);\ g(t) \in \partial\varphi(t, u(t)) \text{ a.e. on }]0, T[.$$

Then $u(t) \in \mathrm{Dom}\,\varphi(t, \cdot)$ *for all* $t \in [0, T]$ *and the function* $t \to \varphi(t, u(t))$ *is absolutely continuous on* $[0, T]$. *More specifically the following inequalities hold*

$$(4.3)\qquad \frac{d}{dt}\varphi_\lambda(t, u(t)) \le [k(0) + \rho\,\|\partial\varphi_\lambda(t, u(t))\|]\,|b'(t)| + \langle \partial\varphi_\lambda(t, u(t)), u'(t)\rangle \text{ a.e. on }]0, T[$$

$$(4.4)\qquad \left|\frac{d}{dt}\varphi(t, u(t))\right| \le h(t), \quad \text{for almost all} \quad t \in]0, T[,$$

where

$$h(t) = [k(0) + \rho\,\|g(t)\|]\,|b'(t)| + \|g(t)\|\,\|u'(t)\|.$$

LEMMA 4.2. *Let Conditions (II) be satisfied and let* T *be fixed in* $]0, +\infty[$. *Consider in* $L^2(0, T; H)$ *the multivalued operator*

$$\mathcal{A}u = \{ v \in L^2(0, T; H) ; v(t) \in \partial\varphi(t, u(t)), \text{ a.e. on }]0, T[\} \tag{4.5}$$

Then $\mathcal{A}$ *is maximal monotone and*

$$(\mathcal{A}_\lambda u)(t) = \partial\varphi_\lambda(t, u(t)), \text{ for almost all } t \in]0, T[\text{ and all } \lambda > 0. \tag{4.6}$$

In the sequel, various finite positive constants independent of λ or T will be denoted by the same symbol C.

Proof of Theorem 1. Uniqueness. Let $u_1(t), u_2(t)$ be two solutions of Eq. (3.8). Then $u_1 - u_2$ satisfies

$$u_1(t) - u_2(t) + \int_0^t a(t-s)[A(s)u_1(s) - A(s)u_2(s)]\,ds = 0, \quad 0 \le t < \infty. \tag{4.7}$$

Form the inner product of (4.6) with $A(t)u_1(t) - A(t)u_2(t)$ and integrate over $]0, T[$, $T > 0$. Taking into account Remark 3.1 and the monotonicity of $A(t)$, we have

$$\int_0^T \langle u_1(t) - u_2(t), A(t)u_1(t) - A(t)u_2(t) \rangle\,dt = 0, \quad \text{for any } T > 0. \tag{4.8}$$

Consequently

$$\langle u_1(t) - u_2(t), A(t)u_1(t) - A(t)u_2(t) \rangle = 0, \quad \text{a.e. on }]0, +\infty[.$$

Since $A(t)$ is assumed to be strictly monotone and u_1, u_2 are continuous we conclude that $u_1(t) = u_2(t)$ for all $t \ge 0$.

Existence. For each $\lambda > 0$ consider the approximating equation

$$u_\lambda(t) + \int_0^t a(t-s) A_\lambda(s) u_\lambda(s)\,ds = f(t), \quad 0 \le t < \infty. \tag{4.9}$$

Inasmuch as $A_\lambda(t)$ is Lipschitzian for all $t \ge 0$, with Lipschitz constant equal to $1/\lambda$, it follows easily that Eq. (4.9) has a unique solution $u_\lambda \in W^{1,2}_{loc}(0, \infty; H)$. Differentiating (4.9) then yields

$$u'_\lambda(t) + a(0) A_\lambda(t) u_\lambda(t) + \int_0^t a'(t-s) A_\lambda(s) u_\lambda(s)\,ds = f'(t), \quad \text{a.e. on }]0, \infty[. \tag{4.10}$$

If we multiply (4.10) by $A_\lambda(t)\, u_\lambda(t)$ and use the estimate (4.3), we get

$$(4.11)\qquad \frac{d}{dt}\varphi_\lambda(t, u_\lambda(t)) + a(0)\,\|A_\lambda(t)\,u_\lambda(t)\|^2 \le \|f'(t)\|\,\|A_\lambda(t)\,u_\lambda(t)\| + \|A_\lambda(t)\,u_\lambda(t)\|\int_0^t |a'(t-s)|\,\|A_\lambda(s)\,u_\lambda(s)\|\,ds + [k(0) + \rho\,\|A_\lambda(t)\,u_\lambda(t)\|]\,|b'(t)|, \quad \text{a.e. on }]0, \infty[.$$

Integrate (4.11) over $]0, T[$, where $T > 0$ is such that $a(T) > 0$. Using the fact that the operator $x \to Lx$ defined by $(Lx)(t) = \int_0^t |a'(t-s)|\,x(s)\,ds$ is linear and bounded from $L^2(0, T; R)$ into itself, one obtains

$$(4.12)\qquad \varphi_\lambda(T, u_\lambda(T)) + a(T)\int_0^T \|A_\lambda(t)\,u_\lambda(t)\|^2\,dt \le \varphi_\lambda(0, f(0)) + k(0)\int_0^T |b'(t)|\,dt + \int_0^T \|A_\lambda(t)\,u_\lambda(t)\|\,(\|f'(t)\| + \rho\,|b'(t)|)\,dt.$$

Then by (2.9) and Schwarz's inequality, we have

$$(4.13)\qquad \varphi_\lambda(T, u_\lambda(T)) + \frac{a(T)}{2}\int_0^T \|A_\lambda(t)\,u_\lambda(t)\|^2\,dt \le C.$$

From (2.8), (3.5) and (4.13) we deduce that

$$(4.14)\qquad \langle p, J_\lambda(T)\,u_\lambda(T)\rangle + \frac{\lambda}{2}\|A_\lambda(T)\,u_\lambda(T)\|^2 + \frac{a(T)}{2}\int_0^T \|A_\lambda(t)\,u_\lambda(t)\|^2\,dt \le C, \qquad \text{for any fixed } p \in D.$$

If we can take $p = 0$ we find that $\{A_\lambda(t)\,u_\lambda(t)\,;\,\lambda > 0\}$ is bounded in $L^2(0, T; H)$. Otherwise, use (4.14) to obtain after simple computations

$$(4.15)\qquad \frac{\lambda}{2}(1-\lambda)\,\|A_\lambda(T)\,u_\lambda(T)\|^2 + \frac{a(T)}{2}\int_0^T \|A_\lambda(t)\,u_\lambda(t)\|^2\,dt \le C + \|p\|\,\|u_\lambda(T)\|.$$

By (4.9),

$$\|u_\lambda(T)\| \le \|f(T)\| + \left(\int_0^T |a(t)|^2\,dt\right)^{1/2}\left(\int_0^T \|A_\lambda(t)\,u_\lambda(t)\|^2\,dt\right)^{1/2}.$$

Hence, (4.15) yields

(4.16) $\{A_\lambda(t)\,u_\lambda(t)\}$ and $\{u_\lambda(t)\}$ are bounded in $L^2(0, T; H)$, as $\lambda \to 0$.

We now choose a sequence $\lambda_n \to 0$ as $n \to \infty$ such that $u_{\lambda_n}(t)$ and $A_{\lambda_n}(t)\,u_{\lambda_n}(t)$ converge weakly in $L^2(0, T; H)$ to u and v respectively. Take $\lambda = \lambda_n$; $\lambda = \lambda_m$ in (4.9) and subtract the corresponding equations to obtain

$$(4.17) \qquad u_{\lambda_n}(t) - u_{\lambda_m}(t) + \int_0^t a(t-s)\,(A_{\lambda_n}(s)\,u_{\lambda_n}(s) - A_{\lambda_m}(s)\,u_{\lambda_m}(s))\,ds = 0, \qquad 0 \le t < +\infty.$$

Multiply (4.17) by $A_{\lambda_n}(t)\,u_{\lambda_n}(t) - A_{\lambda_m}(t)\,u_{\lambda_m}(t)$ and integrate over $]0, T[$. By Remark 3.1,

$$(4.18) \qquad \int_0^T \langle A_{\lambda_n}(t)\,u_{\lambda_n}(t) - A_{\lambda_m}(t)\,u_{\lambda_m}(t)\,,\, u_{\lambda_n}(t) - u_{\lambda_m}(t)\rangle\,dt \le 0, \qquad \text{for all } n, m > 0.$$

In view of Lemma 4.2, the last inequality implies (cf. also [2, Proposition 2.5]) that $u(t) \in D(A(t))$ and $v(t) \in A(t)\,u(t)$, a.e. on $]0, T[$.

Putting $\lambda = \lambda_n$ in Eq. (4.9) and letting $\lambda_n \to 0$, we have

$$(4.19) \qquad u(t) + \int_0^t a(t-s)\,A(s)\,u(s)\,ds \ni f(t), \qquad \text{a.e. on }]0, T[.$$

Differentiating (4.9) we see that $u' \in L^2(0, T; H)$. Hence, by Lemma 4.1 $u(t) \in \operatorname{Dom} \varphi(t, \cdot)$, $t \in [0, T]$ and the function $t \to \varphi(t, u(t))$ is absolutely continuous on $[0, T]$.

It remains to show that $u(t)$ can be continued past T. To this end let us consider the equation

$$(4.20) \qquad w(t) + \int_0^t a(t-s)\,A(T+s)\,w(s)\,ds \ni f(t+T) - \int_0^T a(T+t-s)\,A(s)\,u(s)\,ds, \qquad 0 \le t \le T.$$

If we denote

$$\tilde{\varphi}(s, x) = \varphi(T+s, x), \; s \ge 0, \; x \in H,$$

$$f_1(t) = f(t+T) - \int_0^T a(T+t-s)\,A(s)\,u(s)\,ds, \qquad t \ge 0,$$

then Eq. (4.20) can be rewritten as

$$(4.21) \qquad w(t) + \int_0^t a(t-s)\, \partial\tilde{\varphi}(s, w(s))\, ds \ni f_1(t), \qquad 0 \leq t \leq T.$$

Since, as is easily seen, $\tilde{\varphi}$ and f_1 satisfy Conditions (II) and (3.12), there exists a solution $w \in W^{1,2}(0, T; H)$ of (4.21). Consequently, the function

$$\tilde{u}(t) = \begin{cases} u(t), & 0 \leq t \leq T \\ w(t-T), & T < t \leq 2T, \end{cases}$$

satisfies on $[0, 2T]$ the conclusion of Theorem 1. By the same argument one can extend $u(t)$ on $]2T, \infty[$.

Proof of Theorem 2. Starting from Eq. (4.9) and proceeding as in the proof of Theorem 1 we arrive at (4.15). Inasmuch as $a(t) \geq a(\infty) > 0$, it follows that $\{u_\lambda(t)\}$ and $\{A_\lambda(t)\}$ are bounded in $L^2_{loc}(0, \infty; H)$ as $\lambda \to 0$. Therefore, by applying the diagonal process we find a sequence $\lambda_n \to 0$ such that $u_{\lambda_n}(t) \to u(t)$ and $A_{\lambda_n}(t)\, u_{\lambda_n}(t) \to A(t)\, u(t)$, weakly in $L^2(0, T; H)$, for all $T > 0$. Thus we obtain a solution $u(t)$ of Eq. (3.8) which clearly satisfies (3.13) (see Lemma (4.1).

To prove (3.19)-(3.21) notice first that by (3.15) and (3.18), the constant C appearing in (4.13) is independent of T and λ. From (2.8), (4.13) we then conclude that $\varphi(T, J_\lambda(T)\, u_\lambda(T)) \leq C$, for all $T, \lambda > 0$. Hence, by (3.16),

$$(4.22) \qquad \| J_\lambda(t)\, u_\lambda(t) \| \leq C, \qquad t \geq 0, \lambda > 0.$$

Combining (4.13) with (2.8) and (3.5), we have

$$(4.23) \qquad \frac{a(T)}{2} \int_0^T \| A_\lambda(t)\, u_\lambda(t) \|^2\, dt \leq C + \| p \| \, \| J_\lambda(T)\, u_\lambda(T) \| + \varphi^*(T, p)$$

for any $p \in D$.

Now take $p = p_0$ in (4.23) and make use of (3.14), (3.17) and (4.22) to obtain

$$(4.24) \qquad \int_0^T \| A_\lambda(t)\, u_\lambda(t) \|^2\, dt \leq C, \qquad \text{for all} \quad T \geq 0, \lambda > 0.$$

Since $A_{\lambda_n}(t)\, u_{\lambda_n}(t)$ converges weakly in $L^2(0, T; H)$ to $A(t)\, u(t)$, (4.24) implies that $A(t)\, u(t) \in L^2(0, \infty; H)$. Differentiating (3.8) immediately yields $u' \in L^2(0, \infty; H)$, therefore (3.20) is established.

The remaining assertions of the theorem follow from Lemma 4.1. In fact, taking $g(t) = A(t)\, u(t)$ in (4.4) and using (3.15) one finds that $h(t)$

is integrable over $]0, +\infty[$; hence,

$$|\varphi(t, u(t))| \leq |\varphi(0, f(0))| + \int_0^\infty |h(t)|\, dt = C, \qquad t \geq 0 \tag{4.25}$$

Consequently, by (3.16), $\|u(t)\|$ is bounded on $[0, +\infty[$.

To deduce (3.21) observe that $\int_0^\infty \left|\frac{d}{dt}\varphi(t, u(t))\right| dt < +\infty$. This completes the proof of Theorem. 2.

Proof of Theorem 3. Let us again consider Eq. (4.9) and derive (4.11). If we next integrate (4.11) over $]0, T[$, T arbitrary in $]0, \infty[$ and use (3.15), (3.23) we get the estimate (4.13), where C does not depend on T and λ. Since, by hypothesis $\varphi_\lambda(T, u_\lambda(T)) \geq \varphi(T, J_\lambda(T) u_\lambda(T)) \geq -M$, it follows that

$$\frac{a(T)}{2}\int_0^T \|A_\lambda(t) u_\lambda(t)\|^2\, dt \leq C, \qquad \text{for all } T, \lambda > 0. \tag{4.26}$$

Noting that $a(t) \geq a(\infty) > 0, t \geq 0$ we infer from (4.26) that $\{A_\lambda(t) u_\lambda(t); \lambda > 0\}$ is bounded in $L^2(0, \infty; H)$. Then, use (4.9) and (3.22) in conjunction with (3.23) to obtain

$$\|u_\lambda\|_{L^2(0,\infty;H)} \leq \|f\|_{L^2(0,\infty;H)} + \left(\int_0^\infty |a(t)|\, dt\right) \|A_\lambda u_\lambda\|_{L^2(0,\infty;H)}. \tag{4.27}$$

Therefore $\{u_\lambda; \lambda > 0\}$ is bounded in $L^2(0, \infty; H)$. Choosing a sequence $\lambda_n \to 0$ as $n \to \infty$, we have $u_{\lambda_n}(t) \to u(t)$, $A_{\lambda_n}(t) u_{\lambda_n} \to v(t)$, weakly in $L^2(0, \infty; H)$. It is immediate (see the proof of Theorem 1) that $v(t) \in A(t) u(t)$, a.e. on $]0, +\infty[$ and $u(t)$ satisfies Eq. (3.8).

Further, differentiating (3.8) gives $u' \in L^2(0, \infty; H)$, hence the assertion (3.24) follows. Finally, in view of Lemma 4.1, $\varphi(t, u(t))$ is locally absolutely continuous. Then by (4.4), (3.15) and (3.24),

$$\int_0^\infty \left|\frac{d}{dt}\varphi(t, u(t))\right| dt \leq \int_0^\infty |h(t)|\, dt < +\infty.$$

This implies (3.21) and the proof is complete.

Remark 4.1. Theorem 3 remains valid if we drop (3.17) and strengthen (3.23) to

$$f \in W^{1,2}(0, \infty; H); \qquad \|f\| \text{ bounded on } [0, \infty[. \tag{3.23'}$$

To verify this, use (4.15) and derive the boundedness of $\{A_\lambda(t) u_\lambda(t)\}$ in $L^2(0, \infty; H)$, as $\lambda \to 0$.

Proof of the Corollary. It suffices to notice that Eq. (3.29) is equivalent to

$$u(t)+\int_0^t a_1(t-s)\,\partial\varphi(s,u(s))\,ds \ni f_1(t), \qquad 0\le t<\infty, \tag{3.28}$$

where $a_1(t)=m+\int_0^t a(s)\,ds$ and $f_1(t)=u_0+\int_0^t f(s)\,ds$ satisfy the conditions of Theorem 1 (Theorem 2).

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