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James L. Reid

An N-Dimensional Thomas-Fermi Equation

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Equazioni a derivate parziali. — *An* N-*Dimensional Thomas-Fermi Equation*. Nota di JAMES L. REID, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Si scrive un'equazione, dandone una soluzione particolare, a derivate parziali del secondo ordine, in n variabili indipendenti, che si riduce all'equazione Thomas-Fermi, per $n = 1$.

In a recent Note [1] the Author developed the nonlinear differential equation

$$y'' + r(x)y' + kq(x)y = (1 - l)y'^2 y^{-1} - b^2 u^{2j-2} v^{2n-1} W^2 y^{1-2ml}, \tag{1}$$

which has the homogeneous solution

$$y = \left[\frac{mbu^j v^n}{\pm (kjn)^{1/2}}\right]^{k/m}, \qquad j + n = m, \qquad kl = 1, \tag{2}$$

where b is a constant and the exponents are real and non-zero. The functions $u(x)$ and $v(x)$ are linearly independent solutions of the base equation

$$y'' + r(x)y' + q(x)y = 0, \tag{3}$$

and have the Wronskian $W(x) = uv' - vu' \neq 0$. By choosing the solutions $u = x$ and $v = 1$ of the simple base equation $y'' = 0$ and by selecting $l = 1$, $j = 3/4$, $m = -1/4$, and $b = 1$, one can reduce (1) to the Thomas-Fermi equation

$$y'' = x^{-1/2} y^{3/2}, \tag{4}$$

with the particular solution

$$y = 144\, x^{-3}. \tag{5}$$

The object of this short Note is to present a Thomas-Fermi equation for N independent variables. It is not difficult to extend (1) to N variables $(x_1, \cdots, x_N)$; the procedure is analogous to that followed in [1]. One obtains the form

$$\sum_{i,j=1}^{N} f_{ij} \frac{\partial^2 Y}{\partial x_j \partial x_i} + \sum_{i=1}^{N} g_i \frac{\partial Y}{\partial x_i} + khY = \tag{6}$$

$$= (1 - l) Y^{-1} \sum_{i=1}^{N} f_{ij} \frac{\partial Y}{\partial x_i} \frac{\partial Y}{\partial x_j} - b^2 U^{2j-2} V^{2n-2} W^2 Y^{1-2ml},$$

(*) Nella seduta dell'11 giugno 1975.

where

(7)
$$W^2 = \sum_{i,j=1}^{N} f_{ij} \left[U^2 \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} - \right.$$
$$\left. - UV \left(\frac{\partial U}{\partial x_i} \frac{\partial V}{\partial x_j} + \frac{\partial U}{\partial x_j} \frac{\partial V}{\partial x_i} \right) + V^2 \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \right]$$

is a formal generalization [2] of the Wronskian for second order ordinary differential equations. The coefficients f_{ij}, g_i, and h are functions of $(x_1, \cdots, x_N)$. Functions U and V satisfy the linear base equation.

$$\sum_{i,j=1}^{N} f_{ij} \frac{\partial^2 Y}{\partial x_i \partial x_j} + \sum_{i=1}^{N} g_i \frac{\partial Y}{\partial x_i} + hY = 0 .$$

Now assume $g_i = 0$, $h = 0$, and $f_{ij} = c_{ij} =$ constants, and consider the base equation

(8)
$$\sum_{i,j=1}^{N} c_{ij} \, \partial^2 Y / \partial x_j \, \partial x_i = 0 .$$

The last equation is satisfied by the functions

(9)
$$U = \sum_{i=1}^{N} a_i x_i , \qquad V = 1 ,$$

where the a_i are constants. With these solutions (7) is a constant $W = \omega$, say, such that

(10)
$$\omega^2 = \sum_{i,j=1}^{N} c_{ij} \, a_i \, a_j .$$

To avoid the condition $W = 0$, it is assumed that the sum of constants in (10) does not vanish. If $l = 1$ and if b is defined as

(11)
$$b^2 = [j(j-m)\,\omega]^{-1},$$

it is then possible to put (6) in the form

(12)
$$\sum_{i,j=1}^{N} c_{ij} \, \partial^2 Y / \partial x_j \, \partial x_i = \left(\sum_{i=1}^{N} a_i x_i \right)^{2j-2} Y^{1-2m},$$

satisfied by

(13)
$$Y = \left[\frac{\pm m}{(j^2 - jm)^{1/2} \, \omega} \right]^{1/m} \left(\sum_{i=1}^{N} a_i x_i \right)^{j/m} .$$

For the additional specifications $j = 3/4$ and $m = -1/4$ in (12), one obtains the " N-dimensional " Thomas-Fermi equation

(14)
$$\sum_{i,j=1}^{N} c_{ij} \, \partial^2 Y / \partial x_j \, \partial x_i = \left(\sum_{i=1}^{N} a_i x_i \right)^{-1/2} Y^{3/2},$$

having the particular solution

(15) $$Y = 144\,\omega^2 \left(\sum_{i=1}^{N} a_i x_i\right)^{-3}.$$

In analogy with the one-dimensional case, (14) is invariant under the change of variables

(16) $$Y = \alpha\psi(x_1, \cdots, x_N) \quad , \quad x_i = \beta t_i ,$$

provided that $\alpha\beta^3 = 1$, where α and β are constants.

Finally, if (12) is compared with

(17) $$\sum_{i,j=1}^{N} c_{ij}\, \partial^2 Y / \partial x_j\, \partial x_i = \left(\sum_{i=1}^{N} a_i x_i\right)^{1-M} Y^M ,$$

it is seen that $j = (3 - M)/3$ and $m = (1 - M)/2$; thus (12) yields an "N-dimensional" Emden equation.

The analogies between these equations and their well known one-dimensional forms are rather striking, and it is hoped that the formal results presented here will stimulate interest in the properties of these differential equations.

REFERENCES

[1] J. L. REID (1972) – *Solution to a nonlinear differential equation with application to Thomas-Fermi equations*, « Rend. Accad. Naz. dei Lincei, Cl. Sci. fis. mat. e nat. », *53* (8), 376–379.

[2] J. L. REID and P. B. BURT (1974) – *Solution of nonlinear partial differential equations from base equations*, « J. Math. Anal. Appl. », *47*, 520–530.