Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the solutions of certainnon-self-adjoint differential equations of fourth order

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.6, p. 851–860.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_6_851_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Equazioni differenziali ordinarie. — *On the solutions of certain non-self-adjoint differential equations of fourth order.* Nota di WILLIE E. TAYLOR JR., presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Con opportune ipotesi su $p(x)$ si studia il comportamento asintotico degli integrali dell'equazione $y^{(4)} + (p(x)y)' + p(x)y' = 0$ e della sua equazione aggiunta.

1. INTRODUCTION

This paper is concerned with the solutions of the fourth order linear differential equation

$$y'''' + (p(x)y)' + p(x)y' = 0 \tag{L}$$

where $p(x)$ is a positive continuous function on the interval $[0, \infty)$.

The adjoint equation of (L) is

$$z'''' - (p(x)z)' - p(x)z' = 0. \tag{L*}$$

Note that (L) = (L*) if and only if $p(x) \equiv 0$. Thus the positivity of $p(x)$ implies (L) is non-self-adjoint.

The oscillatory properties of non self-adjoint fourth order differential equations, had not been studied extensively until M. Keener [3] in a recent article considered a class of non-self-adjoint equations which satisfy various disconjugacy conditions.

For completeness, we make the following definitions.

DEFINITION. The statement that a nontrivial solution $y(x)$ of (L) is oscillatory means that $y(x)$ has infinitely many zeros on $[0, \infty)$.

DEFINITION. A nontrivial solution of (L) has an $i_1 - i_2 - \cdots i_n$ distribution of zeros on an interval I, provided there exist points $x_1 < x_2 < \cdots x_n$ in I such that $y(x)$ has a zero at x_k of multiplicity at least i_k for $k = 1, 2, 3, \cdots, n$.

DEFINITION. For equation (L) and $t \in (0, \infty)$, $r_{i_1 i_2 \cdots i_n}(t)$ is the infimum of the numbers $b > t$ such that there exists a nontrivial solution of (L) having an $i_1 - i_2 - \cdots - i_n$ distribution of zeros on $[t, b]$. If no such number b exists, we write

$$r_{i_1 i_2 \cdots i_n}(t) = \infty. \tag{1.1}$$

(*) Nella seduta dell'11 giugno 1975.

If (1.1) holds for each $t \in (0, \infty)$, we write

$$r_{i_1 i_2 \cdots i_n} = \infty.$$

If $i_1 + i_2 = 4$ and $r_{i_1 i_2} = \infty$, then (L) is said to be $i_1 - i_2$ disconjugate. Similar definitions can be made for (L^*).

Keener's work centered on equations for which $r_{22} = r_{13} = \infty$ on $[0, \infty)$ (Class K_I) and those for which $r_{22} = r_{31} = \infty$ on $[0, \infty)$ (Class K_{II}). Using some results of Peterson [5], Keener also concluded that nontrivial solutions of Class K_I equations and Class K_{II} equations fail to have 1–1–2 and 2–1–1 distributions of zeros, respectively. An immediate consequence is that the zeros of an oscillatory solution of a Class K_{II} equation are all simple (multiplicity one).

Ass a notational convenience we introduce the following differential operators:

$$D_3 y = y''' + p(x) y$$

$$D_3^* z = z''' - p(x) z.$$

2. Preliminary Results

In his work, Keener used the equations

$$y'''' + y' = 0 \tag{2.1}$$

and

$$z'''' - z' = 0 \tag{2.2}$$

as his models. For $p(x) \equiv \frac{1}{2}$, (L) and (L^*) reduces to (2.1) and (2.2), respectively.

Our first result will be essential in what follows.

Lemma 2.1. *Let $y(x)$ be a solution of (L), then the functional*

$$F[y(x)] = y(x) D_3 y(x) - y'(x) y''(x)$$

is non-increasing on $[0, \infty)$. Moreover, $F[y(x)] \equiv 0$ on $[a, \infty)$ for some $a \geq 0$ iff $y(x) \equiv 0$.

Proof. Upon differentiating $F[y(x)]$ we find $F'[y(x)] = -y''^2(x) \leq 0$, from which the first part of our theorem follows.

To prove the remaining part of our theorem, note that $F[y(x)] \equiv 0$ on $[a, \infty)$ implies $F'[y(x)] = -y''^2(x) \equiv 0$ on this same interval, from which it follows that $y''(x) \equiv 0$ for $x \geq a$. Thus $y(x) = mx + b$, for some constants m and b. However, if either $m \neq 0$ or $b \neq 0$, $F[y(x)] = p(x) y^2(x) > 0$, contradicting $F[y(x)] \equiv 0$. Consequently, $F[y(x)] \equiv 0$ implies $y(x) \equiv 0$. The other implication follows easily.

COROLLARY 2.2. *If* $y(x)$ *is a nontrivial solution of* (L) *then* $y(x)$ *has at most one double zero, i.e.,* (L) *is* 2–2 *disconjugate on* $[0, \infty)$.

It should be noted that proof of Lemma 2.1 does not depend on the sign of $p(x)$. However, $p(x) > 0$ plays an important role in the proof of our next Lemma. The proof of this lemma is essentially the same as the one given in Lazer's paper [4], for this reason the proof is omitted.

LEMMA 2.3. *Let* $y(x)$ *be a nontrivial solution of* (L) *satisfying* $y(a) \geq 0$, $y'(a) \leq 0$, $y''(a) \geq 0$ *and* $D_3 y(a) \leq 0$ *for some* $a > 0$. *Then* $y(x) > 0$, $y'(x) < 0$, $y''(x) > 0$ *and* $D_3 y(x) < 0$ *on* $[0, a)$.

From Lemma 2.3 we see that a nontrivial solution having a triple zero at $x = a$ cannot vanish to the left of $x = a$. Thus (L) is 1–3 disconjugate on $[0, \infty)$. From our beginning statements it follows that (L) is a Class K_I equation. Moreover, using techniques similar to those of Hanan [2] or Peterson [5] we can show that (L*) is a Class K_{II} equation. We record these facts in our first theorem.

THEOREM 2.4. *Equation* (L) *belongs to Class* K_I *and equation* (L*) *belongs to Class* K_{II}.

The following pair of theorems are due to Keener and are given for reference purposes.

THEOREM 2.5. *If a Class* K_I *equation is oscillatory, then any nontrivial solution having two zeros (counting multiplicities) is oscillatory.*

THEOREM 2.6. *If a Class* K_I *equation is oscillatory, then, given* $a \in [0, \infty)$, *there is a nonoscillatory solution* $y^*(x, a)$ *which vanishes at* $x = a$.

For (2.1), $y^*(x, a) = (e^{-x} - e^{-a})$, however, we also note that equation (2.1) has solutions with no zeros, e.g., $y(x) \equiv 1$. Our next theorem establishes the existence of such a solution for (L). Thus (L) always has a pair of linearly independent nonoscillatory solutions.

THEOREM 2.7. *There exists a non-vanishing solution of* (L).

Proof. Let $u(x), v(x), w(x), z(x)$ be a basis for the solution space of L. Let $y_n(x) = c_{1n} u(x) + c_{2n} v(x) + c_{3n} w(x) + c_{4n} z(x)$ be a solution satisfying

$$y_n(n) = y_n'(n) = y_n''(n) = 0, y_n'''(n) < 0$$

and where $c_{1n}^2 + c_{2n}^2 + c_{3n}^2 + c_{4n}^2 = 1$. Suppose further, without loss of generality, that $\lim c_{in} = c_i$ for $i = 1, 2, 3, 4$. Let

$$y(x) = c_1 u(x) + c_2 v(x) + c_3 w(x) + c_4 z(x).$$

Since $\{y_n(x)\}$ converges uniformly to $y(x)$ on any compact subset of $[0, \infty)$, and $y_n(x) > 0$ on $(0, n)$, we have $y(x) \geq 0$ on $(0, \infty)$. Now $y(x) \not\equiv 0$ since

$$c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1.$$

Furthermore, if there is a point x_0 such that $y(x_0) = 0$, then $y(x)$ has a double zero at x_0 and therefore $y(x)$ oscillates. But this is clearly impossible since $r_{22} = \infty$. Consequently, $y(x) \neq 0$ for all x.

We now state some results concerning (L^*). Since (L^*) belongs to Class K_{II}, the proof of these results can be modeled after Keener's work.

THEOREM 2.8. *Suppose* (L^*) *is oscillatory. Then* (L^*) *has two linearly independent oscillatory solutions and every linear combination of them is oscillatory.*

From Theorem 2.5, we see that if (L) has an oscillatory solution, then any solution having a multiple zero must oscillate. The opposite is true for (L^*), i.e., any nontrivial solution of (L^*) having a multiple zero is nonoscillatory.

We conclude this section by exhibiting the LaGrange bilinear concomitant for solutions of (L) and (L^*).

THEOREM 2.9. *If* $y(x)$ *and* $z(x)$ *are solutions of* (L) *and* (L^*), *respectively, then*

$$J[y, z] = y(x) D_3^* z(x) - y'(x) z''(x) + y''(x) \cdot z'(x) - D_3 y(x) \cdot z(x)$$

is a constant, which is determined by the initial conditions of $y(x)$ *and* $z(x)$.

Let S and S^* denote the solution spaces of (L) and (L^*), respectively. Following Dolan [1], we let $S^*(y) = \{z \in S^* : J[y, z] = 0\}$ where y is a fixed solution in S. It is easy to verify that $S^*(y)$ is a three-dimensional subspace of S^* whenever y not the trivial solution of S. In a similar manner, for a fixed $z \in S^*$, $S(z) = \{y \in S : J[y, z] = 0\}$.

3. Oscillation Properties of (L) and (L^*)

We assume for the remainder of this work that $p(x)$ satisfies the additional hypothesis

$$\text{(H)} \qquad \int_0^\infty p(x)\,dx = \infty.$$

The first result of this section shows that (H) is sufficient for oscillation.

THEOREM 3.1. *Let* $y(x)$ *be a solution of* (L). *Then* $y(x)$ *is oscillatory if and only if* $F[y(x)] < 0$ *on* $[c, \infty)$ *for some* $c \geq 0$.

Proof. Suppose $F[y(c)] < 0$ and that $y(x)$ is nonoscillatory. Then there exists $a \geq c$ such that $y(x) \neq 0$, for $x \geq a$. We can suppose, without loss of generality, that $y(x) > 0$ on $[a, \infty)$.

Consider the function

$$\mathrm{H}(x) = \frac{y''(x)}{y(x)} + \int_a^x p(t)\,\mathrm{d}t\,.$$

By differentiating $\mathrm{H}(x)$ we obtain

$$\mathrm{H}'(x) = \frac{\mathrm{F}[y(x)]}{y^2(x)} < 0\,.$$

Consequently, $\mathrm{H}(x)$ is decreasing. Since $\int_a^\infty p(t)\,\mathrm{d}t = \infty$, it follows that $y''(x) < 0$ for large x and since $y(x) > 0$ it follows that $y'(x) > 0$ for large x.

The fact that $\int_a^x p(t)\,\mathrm{d}t \to \infty$ as $x \to \infty$ implies $\frac{y''(x)}{y(x)} \to -\infty$ as $x \to \infty$. But this implies $y''(x)$ is bounded away from zero for large x, implying $y(x) \to -\infty$ as $x \to \infty$. This contradiction establishes the first part of our theorem.

Now suppose $\mathrm{F}[y(x)] \geq 0$ on $[0, \infty)$. We will show that $y(x)$ is non-scillatory.

The functional

$$\mathrm{G}[z(x)] = z(x)\,\mathrm{D}_3^* z(x) - z'(x)\,z''(x)$$

is decreasing whenever $z(x)$ is a solution of (L^*). This fact is easily verified by computing $\mathrm{G}'[z(x)]$.

Suppose $z(x) \in \mathrm{S}^*(y)$, $\mathrm{G}[z(x)] < 0$ for large x and $z(x) > 0$ for large x. Such a solution $z(x)$ exists because $\mathrm{S}^*(y)$ is a 3-dimensional subspace of S^*, and consequently, some nontrivial $z \in \mathrm{S}^*(y)$ has a double zero. Recall that (L^*) is of Class K_{II} and hence solutions with double zeros do not oscillate.

Now suppose $y(x)$ is oscillatory. If x_0 is a zero of $y(x)$ it follows that $y'(x_0) \neq 0$ and $y''(x_0) \neq 0$. For if so, then $\mathrm{F}[y(x)] \equiv 0$ on $[x_0, \infty)$, implying $y(x) \equiv 0$.

Let $\alpha < \beta$ consecutive zeros of $y(x)$ greater than a and suppose $y(x) > 0$ on (α, β). Differentiating $\frac{y''(x)}{z(x)}$ we get

$$\left(\frac{y''(x)}{z(x)}\right)' = \frac{z(x)\,y'''(x) - z'(x)\,y''(x)}{z^2(x)}\,. \tag{3.1}$$

Since $z(x) \in \mathrm{S}^*(y)$,

$$y(x)\,\mathrm{D}_3 z(x) - y'(x)\,z''(x) + y''(x)\,z'(x) - \mathrm{D}_3 y(x)\,z(x) = 0$$

and so

$$z(x)\,y'''(x) - z'(x)\,y''(x) = y(x)\,\mathrm{D}_3 z(x) - y'(x)\,z''(x) - p(x)\,y(x)\,z(x)$$

for all x. Substituting in (3.1) we obtain

$$\left(\frac{y''(x)}{z(x)}\right)' = \frac{D_3 z(x) - z''(x) y'(x) - p(x) z(x) y(x)}{z^2(x)} . \tag{3.2}$$

Integrating (3.2) by parts yields

$$\frac{y''(\beta)}{z(\beta)} - \frac{y''(\alpha)}{z(\alpha)} = \int_\alpha^\beta 2 \frac{G[z(x)]}{z^3(x)} y(x)\, dx . \tag{3.3}$$

Since $F[y(x)] > 0$ for all x, at a zero of $y(x)$, say x_0, $\operatorname{sgn} y''(x_0) \neq \operatorname{sgn} y'(x_0)$. From this observation and the fact that $y(x) > 0$ on (α, β) we conclude that the left side of (3.3) is positive and the right side of (3.3) is negative. This proves that $y(x)$ cannot oscillate if $F[y(x)] > 0$ for all x.

THEOREM 3.4. *Let $y(x)$ be a nonoscillatory solution of* (L), *then*

$$\int_a^\infty y''^2(x)\, dx < \infty .$$

Proof. From Theorem 3.2 it follows that $F[y(x)] > 0$ for all $x \in [0, \infty)$. Differentiating $F[y(x)]$ and integrating from zero to x we obtain the following inequality

$$0 < F[y(x)] = F[y(0)] - \int_0^x y''^2(t)\, dt .$$

Since x arbitrary, we obtain

$$\int_0^\infty y''^2(x)\, dx < F[y(0)] ,$$

which completes our proof.

Before we investigate the behavior of the oscillatory solutions of (L), a lemma will be needed.

LEMMA 3.5. *Let $y(x)$ be an oscillatory solution of* (L). *Then the functional* $N[y(x)] = y(x) y''(x) - \frac{1}{2} y'^2(x) \to -\infty$ *as* $x \to \infty$.

Proof. Note that $N'[y(x)] = F[y(x)] - p(x) y^2(x) \leq F[y(x)]$. Since $y(x)$ is oscillatory, it follows that $F[y(x)]$ is negative and bounded away from zero for large x. Thus, $N[y(x)] \to -\infty$ as $x \to \infty$.

THEOREM 3.6. *Let $y(x)$ be an oscillatory solution of* (L). *Then*

(i) $y'(x)$ *is unbounded*;

(ii) $\displaystyle\int_a^\infty y'^2(x)\, dx = \infty$, *and*

(iii) $y''(x)$ *is unbounded.*

Proof. From the preceding lemma the functional

$$\mathrm{N}[y(x)] = y(x)\,y''(x) - \tfrac{1}{2}y'^2(x) \to -\infty \qquad \text{as} \quad x \to \infty,$$

from which (i) follows easily.

To prove (ii) we integrate N $[x]$ from c to x where $y'(c) = 0$. Doing so we obtain

$$\int_c^x \mathrm{N}[y(t)]\,dt = y(x)\,y'(x) - \frac{3}{2}\int_c^x y'^2(t)\,dt.$$

But $\int_c^x \mathrm{N}[y(t)]\,dt \to -\infty$ as $x \to \infty$ and since $y(x)$ oscillates (ii) follows.

Multiplying (L) by $y'(x)$ and integrating yields

$$y'(x)\,\mathrm{D}_3 y(x) - \tfrac{1}{2}y''^2(x) = k + \int_c^x p(t)\,[y(t)\,y''(t) - y^{2\prime}(t)]\,dt.$$

As $x \to \infty, \int_c^x p(t)\,[y(t)\,y''(t) - y'^2(t)]\,dt \to \infty$ and therefore $y''^2(x) \to \infty$ along the zeros of $y'(x)$. Thus (iii) holds.

By Theorem 3.4, the second derivative of a nonoscillatory solution to (L) is square-integrable. We also see from Theorem 3.6, that the second derivative of an oscillatory solution is unbounded. Thus, we ask " are the nonoscillatory solutions of (L) precisely the solutions having a square-integrable second-derivative? " This question remains open. However, we offer the following result.

THEOREM 3.7. *Let $y(x)$ be an oscillatory solution of* (L). *If $p'(x) \leq 0$, then* $\int_0^\infty y''^2(x)\,dx = \infty.$

Proof. In the proof of Theorem 3.6, we found that

$$\mathrm{Q}[y(x)] = y'(x)\,\mathrm{D}_3 y(x) - y''^2(x) \to -\infty \tag{3.4}$$

as $x \to \infty$.

Integrating Q $[y(x)]$ from zero to x we get

$$y'(x)\,y''(x) - \int_0^x y''^2(t)\,dt + \tfrac{1}{2}p(x)\,y^2 - \tag{3.5}$$

$$-\tfrac{1}{2}\int_0^x p'(t)\,y^2(t)\,dt = k + \int_0^x \mathrm{Q}(t)\,dt.$$

As $x \to \infty$, the right side and hence the left side of (3.5) tends to $-\infty$. But this implies $\int\limits_0^x y''^2(t)\,dt \to \infty$ as $x \to \infty$.

Our next result follows from Theorems 3.4, 3.6 and 3.7.

THEOREM 3.8. *Let* $y(x)$ *be a solution of* (L). *If* $p'(x) \leq 0$, *then these are equivalent*:

(i) $y(x)$ *is nonoscillatory*;

(ii) $F[y(x)] \geq 0$;

(iii) $\int\limits_0^\infty y''^2(x)\,dx < \infty$.

Examining equation (2.1), the reader may note that the nonoscillatory solutions of (2.1) form a two-dimensional subspace. The question that presents itself is whether or not a similar statement can be made concerning the non-oscillatory solutions of (L). This remains an open question. However; a result in this direction is given below.

THEOREM 3.9. *If* $p'(x) \leq 0$, *then the set of nonoscillatory solutions of* (L) *together with the trivial solution form a 2-dimensional subspace of* S.

Proof. Let $y^*(x, a)$ and $w(x)$ be two linearly independent nonoscillatory solutions whose existence was discussed in Theorems 2.6 and 2.7. Suppose $u(x)$ is a nonoscillatory solution of (L) which is independent of $y^*(x, a)$ and $w(x)$, then some nontrivial linear combination $v(x)$ of $u(x)$, $y^*(x, a)$ and $w(x)$ has two zeros and hence is oscillatory. Since $p'(x) \leq 0$, we have $\int\limits_0^\infty v''^2(x)\,dx = \infty$. But each of $u(x)$, $y^*(x, a)$ and $w(x)$ have a square-integrable second derivative and thus

$$v''(x) = c_1 u''(x) + c_2 y^{*''}(x, a) + c_3 w''(x)$$

is square-integrable, a contradiction.

An immediate consequence of Theorem 3.9 is that $y^*(x, a)$ is essentially unique, i.e., $y^*(x, a)$ is the only nonoscillatory solution (except for scalar multiples) which vanishes at $x = a$. Also from Theorem 3.9 we see that the sum of an oscillatory solution and a nonoscillatory solution is oscillatory. We use this fact in our next proof.

THEOREM 3.10. *Suppose* $p'(x) \leq 0$. *If* (L) *has a nonoscillatory solution* $y(x)$ *such that*

$$\liminf_{x\to\infty} |y(x)| \neq 0$$

then the oscillatory solutions of (L) *are unbounded.*

Proof. Let $u(x)$ be an oscillatory solution of (L). Then $s(x) = u(x) - ky(x)$ is oscillatory for all $k > 0$. Since $\liminf_{x\to\infty} |y(x)| \neq 0$, we can suppose $\liminf_{x\to\infty} |y(x)| = 1$, but $s(x)$ oscillatory implies $u(x)$ intersects $ky(x)$ for each $k > 0$ and therefore $\limsup_{x\to\infty} u(x) = \infty$.

Before we prove the existence of oscillatory solutions for (L^*) we need the following lemmas. Their proofs are similar to the proofs of Lemma 2.1 and Theorem 2.7.

LEMMA 3.11. *Let $z(x)$ be a solution of (L^*). Then the functional* $G[z(x)] = z(x)\, D_3^* z - z'(x)\, D_2^* z$ *is nonincreasing on* $[0, \infty)$.

LEMMA 3.12. *Given $b \in [0, \infty)$, there exists a solution $z(x)$ of (L^*) such that $z(b) = 0$ and $G[z(x)] > 0$ for all $x \in [0, \infty)$.*

We proceed to show that (L^*) is oscillatory whenever (L) is oscillatory.

THEOREM 3.13. *Let $z(x)$ be a solution of (L^*) such that $G[z(x)] > 0$ for all $x \geq 0$, then $z(x)$ is oscillatory.*

Proof. Suppose $z(t)$ does not oscillate. Assume, without loss of generality that $z(a) = 0$ and $z(x) > 0$ for all $x > a$ for some a.

Let $y(x)$ be the solution of (L) satisfying $y(a) = y'(a) = y''(a) = 0$, $y'''(a) = 1$, then $y(x)$ is oscillatory by Theorem 3.1. Let α and β $(a < \alpha < \beta)$ be consecutive zeros of $y(x)$ where $y(x) > 0$ and (α, β).

Consider the following identity.

$$\left(\frac{y''}{z}\right)' = \frac{zy''' - z'y''}{z^2}. \tag{3.6}$$

From the initial conditions of $y(x)$ and $z(x)$

$$J[y, z] = y(x)\, D_3^* z(x) - y'(x)\, z''(x) + y''(x)\, z'(x) - D_3 y(x)\, z(x) \equiv 0.$$

Thus

$$z(x)\, y'''(x) - z'(x)\, y''(x) = y(x)\, D_3^* z(x) - y'(x)\, z''(x) - p(x)\, z(x)\, y(x).$$

Substituting on the right side of (3.6) we get

$$\left(\frac{y''}{z}\right)' \frac{yD_3^* z - y'z'' - pyz}{z^2}. \tag{3.7}$$

Integrating (3.7) from α to β, yields

$$\frac{y''(\beta)}{z(\beta)} - \frac{y''(\alpha)}{z(\alpha)} = \int_\alpha^\beta \frac{2y(x)\, G[z(x)]}{z^3(x)}\, dx. \tag{3.8}$$

Since $y(x) > 0$ on (α, β) and $F[y(\alpha)] = -y'(\alpha)\, y''(\alpha) < 0$ and $F[y(\beta)] = -y'(\beta)\, y''(\beta) < 0$, we conclude that $y'(\alpha) > 0$, $y''(\alpha) > 0$, $y'(\beta) < 0$, and $y''(\beta) < 0$. This implies that the left side of (3.8) is negative, however, the right side of (3.8) is positive. This contradiction proves that $z(x)$ must oscillate.

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