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B. E. Rhoades

A class of generalized-Hilbert-Schmidt operators

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Analisi funzionale. — *A class of generalized-Hilbert-Schmidt operators.* Nota di B. E. RHOADES, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — G. H. Constantin ha definito una classe di operatori di Cesàro–Hilbert–Schmidt. In questa Nota l'Autore trova la corrispondente proprietà per una più generale classe di operatori di Hilbert–Schmidt (G. H. S.).

Let $\mathscr{A}$ denote the set of infinite matrices with the following properties:

(i) $A \in B(l^2)$,

(ii) $A \notin B(l)$,

(iii) $a_{nk} \geq 0$ for each n and k,

(iv) $t_n = \sum_{k=1}^{\infty} a_{nk} \geq \delta > 0$ for each n.

DEFINITION 1. *Let* $T \in B(H_1, H_2)$, H_i *separable Hilbert spaces.* T *is called generalized-Hilbert-Schmidt if, for every orthonormal basis* $\{e_n\}$ *of* H_1, *and a fixed* $A \in \mathscr{A}$,

$$\sup_k \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \| Te_j \| \right)^2 < \infty. \tag{1}$$

In [3] it was shown that C_α, the Cesàro methods of order $\alpha > 0$, and the Γ_a^α methods, for $\alpha > 0$, $a > \frac{1}{2}$ are bounded operators on l^2. It is well known that $C_\alpha \notin B(l)$, and it can be shown that $\Gamma_a^\alpha \notin B(l)$ for $\mathscr{A} < 1$. $\mathscr{A}$ does not contain all of the Hausdorff matrices in $B(l^2)$, since the Euler methods, $(E, q) \in B(l)$ for $0 < q < 1$. $\mathscr{A}$ also contains many Nörlund and weighted mean methods, as well as numerous other summability methods.

LEMMA 1. *Every Hilbert-Schmidt operator belongs to* GHS.

Proof. It is well-known that $\| T \|_2^2 = \sum_{j=1}^{\infty} \| Te_j \|^2$ for every Hilbert-Schmidt operator T. Thus $\{\|Te_j\|\} \in l^2$. From (i),

$$u_n = \sum_{j=1}^{\infty} a_{nj} \| Te_j \| \in l^2.$$

Therefore

$$\sup_k \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \| Te_j \| \right)^2 = \sup_k \sum_{n=1}^{\infty} a_{nk} u_n^2 = \sup_k \| u \|^2 \sum_{n=1}^{\infty} a_{nk} (u_n / \| u \|)^2$$

$$\leq \sup_k \| u \|^2 \sum_{n=1}^{\infty} a_{nk} | u_n | / \| u \| = \sup_k \| u \| \sum_{n=1}^{\infty} a_{kn}^* | u_n | < \infty.$$

(*) Nella seduta dell'11 giugno 1975.

LEMMA 2. *Every* GHS *operator is compact.*

Proof. From [2], for every orthonormal bases $\{e_n\}$ we have

$$\lim_n \| Te_n \| = 0 .$$

Assume the contrary; i.e., assume there exists an $\varepsilon > 0$ and an orthonormal basis $\{e_n\}$ such that $\| Te_n \| > \varepsilon$ for each n. Then

$$\sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \| Te_j \| \right)^2 > \sum_{n=1}^{\infty} a_{nk} \left(\varepsilon \sum_{j=1}^{\infty} a_{nj} \right)^2 \geq (e\delta)^2 \sum_{n=1}^{\infty} a_{nk} = \infty,$$

a contradiction.

THEOREM 1. *Let* $T \in B(H_1, H_2)$. $T \in GHS$ *if and only if* T *is of the form*

$$(2) \qquad Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle h_n$$

where $\{e_n\}$, $\{h_n\}$ *are orthonormal bases of* H_1, H_2, *respectively, and* $\lambda_n \geq 0$ *such that*

$$(3) \qquad \sup_k \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \lambda_j \right)^2 < \infty.$$

Proof. Let $T \in GHS$. From Lemma 2, T is compact, so it has the form (2), where $\{e_n\}$ are eigenvectors corresponding to the eigenvalues $\{\lambda_n\}$ of the positive compact part S of the polar decomposition $T = US$. Since $Te_k = \lambda_k h_k$, we have, substituting in (1),

$$\infty > \sup_k \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \| Te_j \| \right)^2$$

$$= \sup_k \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} | \lambda_j | \right)^2$$

$$= \sup_k \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \lambda_j \right)^2 .$$

Conversely, suppose T is of the form (2) with $\{\lambda_n\}$ satisfying (3). It follows that $\lambda_n \to 0$. For, if not, then there exists an $\varepsilon > 0$ such that $\lambda_n > \varepsilon$ for all n. Thus

$$\sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \lambda_j \right)^2 > ({}^0\varepsilon)^2 \sum_{n=1}^{\infty} a_{nk} = \infty ,$$

contradicting (3).

T is compact and satisfies (1).

PROPOSITION 1. *Fix* $A \in \mathscr{A}$. *Then the corresponding set of* GHS *operators is a linear space.*

Proof. Let S , T ∈ GHS for the same A. By Minkowski's inequality,

$$\sup_k \left[\sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \| (S+T) e_j \|\right)^2\right]^{1/2}$$

$$\leq \sup_k \left[\sum_{n=1}^{\infty} \left(\sqrt{a_{nk}} \sum_{j=1}^{\infty} \| Se_j \| + \sqrt{a_{nk}} \sum_{j=1}^{\infty} \| Te_j \|\right)^2\right]^{1/2}$$

$$\leq \sup_k \left[\sum_{n=1}^{\infty} a_{nk} \left(\sum_{n=1}^{\infty} \| Se_j\right)^2\right]^{1/2} + \sup_k \left[\sum_{n=1}^{\infty} a_{kn} \left(\sum_{j=1}^{\infty} a_{nj} \| Te_j \|\right)^2\right]^{1/2} < \infty .$$

That T ∈ GHS implies λT ∈ GHS for scalars λ is trivial.

PROPOSITION 2. *Let* S ∈ B (H_2) , T ∈ GHS. *Then* ST ∈ GHS.

Proof.

$$\sup_k \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \| STe_j \|\right)^2 \leq \sup_k \| S \|^2 \sum_{n=1}^{\infty} a_{nk} \left(\sum_{j=1}^{\infty} a_{nj} \| Te_j \|\right)^2 < \infty .$$

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[3] B. E. RHOADES (1971) – *Spectra of some Hausdorff operators*, « Acta Sci. Math. », *32*, 91–100.