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Orthogonality Preserving Operators

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Analisi funzionale. — *Orthogonality Preserving Operators.* Nota I di W. A. AL-SALAM e A. VERMA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Riprendendo un problema considerato da S. Pincherle (1928), da W. Hahn (1949) e successivamente da altri, i due Autori trovano classi di polinomi ortogonali per le quali esiste una trasformazione preservante l'ortogonalità.

1. INTRODUCTION

In this note we shall be concerned with linear operators defined on polynomials by means of

$$\mathrm{J}x^n = \lambda_n x^n \qquad (n = 0, 1, 2, 3, \cdots) \tag{1.1}$$

where $\lambda_0 = 1$, $\lambda_n \neq 0$ for all $n > 0$. Because of the identity $x^n \mathrm{D}^n = x\mathrm{D}(x\mathrm{D} - 1) \cdots (x\mathrm{D} - n + 1)$, the operator (1.1) can be represented by either of the forms

$$\mathrm{J} = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \mathrm{D}^n = \sum_{n=0}^{\infty} \frac{b_n}{n!} (x\mathrm{D})^n \tag{1.2}$$

where $a_0 = b_0 = 1$, and $\lambda_n = \sum_{k=0}^{n} \binom{n}{k} a_k$.

In this note we raise the question: *for what orthogonal polynomial sets (henceforth we indicate by* OPS) $\{\mathrm{P}_n(x)\}$ *and operator* J *of the form* (1.1) *is the polynomial set* $\{\mathrm{Q}_n(x) = \mathrm{JP}_n(x)\}$ *also orthogonal*? For similar problems see [1], [2].

An OPS that we shall encounter is a q-generalization of the Jacobi polynomials which appeared in [3]. They may be defined by

$$\mathrm{J}_n(q, \alpha, \beta; x) = (-1)^n \frac{[\alpha]_n q^{n(n-1)/2}}{[\beta q^{n-1}]_n} {}_2\Phi_1(q^{-n}, \beta q^{n-1}; \alpha; qx) \tag{1.2}$$

where $[\alpha]_0 = 1$, $[\alpha]_n = (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{n-1})$ and ${}_2\Phi_1$ is the Heine series

$${}_2\Phi_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{[a]_n [b]_n}{[q]_n [c]_n} x^n.$$

It is easy to see that as $q \to 1$, ${}_2\Phi_1(a, b; c; x) \to \mathrm{F}(a, b; c; x)$. Hence the q-Jacobi polynomials are proper generalization of the classical orthogonal

(*) Nella seduta del 10 maggio 1975.

polynomials. Hahn [3] gives a three term recurrence relation. We can show that these polynomials are orthogonal with respect to a discrete function $\psi(x)$ which has at the points $x = q^k$ the jump

$$\left\{ \prod_{j=0}^{\infty} \frac{1 - \alpha q^j}{1 - \beta q^j} \right\} \alpha^k \frac{[\beta/\alpha]_k}{[q]_k} \qquad k = 0, 1, 2, \cdots$$

provided that $0 < q < 1$, $0 < \beta < \alpha < 1$ whereas if the value of $q > 1$ and $1 < \alpha < \beta$ then the distribution function $\psi(x)$ is the step function with jump

$$\left\{ \prod_{j=0}^{\infty} \frac{1 - \alpha^{-1} q^{-j}}{1 - \beta^{-1} q^{-j}} \right\} \frac{[\beta/\alpha]_k}{[q]_k}$$

at the points $x = \frac{\alpha}{\beta} q^{-k}$ and $k = 0, 1, 2, \cdots$

2. Necessary and Sufficient conditions

We first recall that the operator (Pincherle [4])

$$J_1 = \lambda^{xD} = \sum_{n=0}^{\infty} \frac{(\lambda - 1)^n}{n!} (xD)^n \tag{2.1}$$

which has the property that $J_1 f(x) = f(\lambda x)$ is of the type (1.1). This operator takes every OPS into an OPS. As we shall see, it is the only one of the type (1.1) which takes every OPS into an OPS.

A slightly less trivial operator is

$$J_2 = A_1 \lambda^{xD} + A_2 (-\lambda)^{xD} \tag{2.2}$$

which clearly satisfies $J_2 f(x) = A_1 f(\lambda x) + A_2 f(-\lambda x)$. If $\{f_n(x)\}$ is a symmetric polynomial set, i.e., $f_n(-x) = (-1)^n f_n(x)$ then $J_2 f_n(x) = (A_1 + (-1)^n A_2) f_n(\lambda x)$. Consequently the operator (2.2) takes every symmetric OPS into an OPS (provided that $A_1^2 \neq A_2^2$). If on the other hand $\{p_n(x)\}$ is not symmetric then we can show that $q_n(x) = J_2 p_n(x)$ is on OPS if and only if A_1 or A_2 (but not both) is zero. The proof of this statement is straightforward but lengthy. We omit it here.

Because of the above discussion we shall refer to the two operators (2.1) and (2.2) as the trivial operators.

Using the method used by Krall and Sheffer in [2] one can prove the following theorem. Since the proof is mere paraphrasing of the proofs of similar theorems given in [2] we omit it here.

Theorem 1. *Let* $\{p_n(x)\}$ *be an* OPS *with associated moments* $\{\alpha_n\}$. *In order that* $\{q_n(x) = J p_n(x)\}$, *where* J *is given by* (1.1), *be also an* OPS *it is necessary and sufficient that there exist constants* $\{\alpha_n^*\}$ *and* $\{A_{ji}^*\}$

such that

(2.3) $$\lambda_n = \sum_{k=0}^{n} \binom{n}{k} a_k \neq 0 \qquad n = 0, 1, 2, \cdots$$

(2.4) $$\sum_{j=0}^{s} \alpha_{n+j} A^*_{s,j} = \alpha^*_{n+s} \lambda_n \qquad \text{for} \quad n, s = 0, 1, 2, 3, \cdots$$

(2.5) $$A^*_{s,s} \neq 0 .$$

When these conditions are satisfied $\{\alpha^*_n\}$ *are moments associated with* $\{q_n(x)\}$.

3. SYMMETRIC POLYNOMIALS

We first note that operators of type (1.1) take every symmetric polynomial set into a symmetric set. It is well known that such sets have odd moments $\alpha_{2n+1} = 0$. On the other hand all even moments are non-zero. We now give

THEOREM 2. *If* $\{p_n(x)\}$ *is a symmetric* OPS *with associated moments* $\{a_n\}$ *and if* J *is a non-trivial operator of the form* (1.1), *and if furthermore* $\{J p_n(x) = q_n(x)\}$ *is also an* OPS *then there are constants* $a, b, \alpha, \beta, \gamma$ *and* q *so that*

(3.1) $$\text{(i)} \quad \alpha_{2n} = a^n [\alpha q]_n / [\beta q]_n , \qquad \text{(ii)} \quad \alpha^*_{2n} = \left(\frac{a}{b}\right)^n [\gamma q]_n / [\beta q]_n$$

and

(3.2) $$\text{(i)} \quad \lambda_{2n} = b^n [\alpha q]_n / [\gamma q]_n , \qquad \text{(ii)} \quad \lambda_{2n+1} = \lambda_1 b^n [\alpha q^2]_n / [\gamma q^2]_n .$$

To prove this theorem we first put $s = 0, 1$ in (2.4) and $\alpha_{2n+1} = 0$ we get $\alpha_n = \lambda_n \alpha^*_n$ and $A^*_{11} = \lambda_{2n+1}/\lambda_{2n+2}$. If then we put $s = 2$ in (2.4) we conclude that $A^*_{21} = 0$ and

(3.3) $$A^*_{20} \alpha_{2n} + \alpha_{2n+2} A^*_{22} = \alpha_{2n+2} \frac{\lambda_{2n}}{\lambda_{2n+2}} .$$

If $A^*_{20} = 0$ we get $\lambda_{2n}/\lambda_{2n+2} = 1/\lambda_2$ which implies that $\lambda_{2n} = \lambda_2^n$ and $\lambda_{2n+1} = A^*_{11} \lambda_2^n$ so that J must be a trivial operator. Hence we may now assume that $A^*_{20} \neq 0$ so that $\lambda_{2n}/\lambda_{2n+2} \neq A^*_{22}$ and we have from (3.3)

(3.4) $$\frac{\alpha_{2n+2}}{\alpha_{2n}} = \frac{A^*_{20}}{\dfrac{\lambda_{2n}}{\lambda_{2n+2}} - A^*_{22}} .$$

Now putting $s = 3$ in (2.4) we get

(3.5) $$A^*_{31} \alpha_{2n} = \alpha_{2n+2} \left(\frac{\lambda_{2n-1}}{\lambda_{2n+2}} - A^*_{33}\right) .$$

Again $A^*_{31} \neq 0$ for otherwise we get (3.5) that $\lambda_{2n}/\lambda_{2n+2} = a$ constant and J is the trivial operator. If $A^*_{31} \neq 0$ but $A^*_{31} A^*_{22} - A^*_{33} A^*_{20} \neq 0$ we again obtain the trivial operator for J. Thus we may further assume $A^*_{31} A^*_{22} - A^*_{33} A^*_{20} = 0$,

$A_{31}^* \neq 0$. Now put $s = 4, 2n$ for n in (2.4) then substitute for $\lambda_{2n}/\lambda_{2n+2}$ from (3.3) and put $\xi = \alpha_{2n+2}/\alpha_{2n}$; we get that ξ_n must satisfy a recurrence relation of the form

$$D\xi_n + \xi_{n+1} + B\xi_n \xi_{n+1} + E = 0 \tag{3.6}$$

whose general solution can be given as

$$\xi_n = \frac{a(1 - \alpha q^n)}{1 - \beta q^n} . \tag{3.7}$$

Actually in finding solutions of (3.6) we disregarded possible cases when $\xi_n = 0$ for all n or for $0 \leq n \leq N$ or for $n > N$ because ξ_n is the ratio of successive moments and thus these cases would lead to Hankel determinants which vanish in violation of well known necessary conditions for $\{\alpha_n\}$ to be a moment sequence.

Now (3.7) implies the validity of (3.1) and (3.2).

The OPS whose moments are given by (3.1–i) is

$$p_{2n}(x) = J_n\left(q, \alpha q, \beta q; \frac{x^2}{a}\right) \quad , \quad p_{2n+1}(x) = x J_n\left(q, \alpha q^2, \beta q^2; \frac{x^2}{a}\right) \tag{3.8}$$

and the transformed OPS $\{q_n(x)\}$ is

$$q_{2n}(x) = \frac{[\alpha q]_n}{[\gamma q]_n} J_n(q, \gamma q, \beta q; bx^2/a) \tag{3.9}$$

$$q_{2n+1}(x) = \lambda_1 \frac{[\alpha q^2]_n}{[\gamma q^2]_n} x J_n(q, \gamma q^2, \beta q^2; bx^2/a).$$

To verify this assertion we make use of the basic analogue of the Saalschutz Theorem for the sum of a terminating ${}_3\varphi_2$. Indeed we have from (3.1) and (1.2)

$$\int_{-\infty}^{\infty} x^{2j} P_{2n}(x)\, d\psi(x) = K_{n,j}\, {}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha q^{j+1}, \beta q^n; \\ \alpha q, \beta q^{j+1}; \end{matrix} \; q \right].$$

The sum of the ${}_3\Phi_2$ in the right hand side [5] is given by

$$\frac{[q^{-j}]_n \left[\frac{\alpha}{\beta} q^{1-n}\right]_n}{[\alpha q]_n \left[\frac{1}{\beta} q^{-j-n}\right]_n}$$

which is zero for $j = 0, 1, \cdots, n-1$. On the other hand

$$\int_{-\infty}^{\infty} x^{2j-1} P_{2n}(x)\, d\psi(x) = 0$$

trivially since all odd moments are zero.

The remaining cases

$$\int_{-\infty}^{\infty} x^j \, P_{2n+1}(x) \, d\psi(x) = 0 \qquad j = 0, 1, \cdots, 2n$$

can be verified similarly.

To verify (3.9) we only need to use (3.2).

4. Now-Symmetric OPS.

Let us now consider the non-symmetric case under the further condition that the operator J in (1.2) is such that $a_2 \neq a_1^2$. This is equivalent to assuming that $\lambda_2 \neq \lambda_1^2$ in (1.1).

As above if we put $s = 0,1$ in (2.4) we obtain

$$\text{(4.1)} \qquad A_{10}\,\alpha_n + A_{11}\,\alpha_{n+1} = \frac{\lambda_n}{\lambda_{n+1}}\,\alpha_{n+1} \qquad n = 0, 1, 2, \cdots$$

Putting $n = 0, 1$ in (4.1) and eliminating A_{11} we get

$$A_{10} = \frac{\alpha_1\,\alpha_2\,(\lambda_2 - \lambda_1^2)}{\lambda_1\,\lambda_2\,(\alpha_2 - \alpha_1^2)} \, .$$

Note that since $\{\alpha_n\}$ is a moment sequence then $\alpha_2 - \alpha_1^2 \neq 0$ and the numerator does not vanish by assumption. Hence $A_{10} \neq 0$. We now can see easily that (4.1) implies that $\alpha_n \neq 0$ for all n.

Now if we put $\frac{\alpha_n}{\alpha_{n+1}} = \xi_n$ we can rewrite (4.1) in the form

$$\text{(4.2)} \qquad A_{10}\,\xi_n + A_{11} = \frac{\lambda_n}{\lambda_{n+1}} \, .$$

Now putting $s = 2$ in (2.4) and using (4.2) we see that $\{\xi_n\}$ satisfies a recurrence relation of the form

$$\xi_n + D_1\,\xi_{n+1} + D_2\,\xi_n\,\xi_{n+1} + D_3 = 0$$

which has the non-trivial general solution

$$\xi_n = \frac{1}{a}\,\frac{1 - \beta q^n}{1 - \alpha q^n}$$

for some constants a, α, β, q.

Hence

$$\text{(4.3)} \qquad \alpha_n = a^n \frac{[\alpha]_n}{[\beta]_n}$$

so that

$$\text{(4.4)} \qquad p_n(x) = a^n\, J_n\left(q, \alpha, \beta; \frac{x}{a}\right)$$

and from (4.2)

$$\lambda_n = b^n \frac{[\alpha]_n}{[\gamma]_n} \tag{4.5}$$

so that

$$q_n(x) = \alpha^n \frac{[\alpha]_n}{[\gamma]_n} J_n\left(q, \gamma, \beta; \frac{bx}{a}\right). \tag{4.6}$$

It is interesting to note that the above results show that the q-generalization of the Jacobi polynomials form the only family of orthogonal polynomial sets for which there exist orthogonality preserving operator of the form (1.2) with $a_1^2 \neq a_2$. This operator is

$$J = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k D^k \qquad (D = d/dx) \tag{4.7}$$

where

$$a_k = \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} \frac{[\alpha]_j}{[\gamma]_j} b^j. \tag{4.8}$$

The requirement $a_1^2 \neq a_2$ is equivalent to $a \neq \gamma$. This restriction can be removed since in this case $J = b^{xD}$ is the trivial operator.

We shall give special cases of these results in our following note.

References

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