Paul Venzke

Finite groups whose maximal subgroups are modular

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Teoria dei gruppi. — *Finite groups whose maximal subgroups are modular.* Nota di PAUL VENZKE, presentata (*) dal Socio G. SCORZA DRAGONI.

RIASSUNTO. — Si indicano altre due caratterizzazioni per i gruppi finiti che hanno come sottogruppi modulari tutti i loro sottogruppi massimi.

The maximal subgroup M of the group G is called *modular* in G whenever the modular law $U \cap \langle V, M \rangle = \langle V, U \cap M \rangle$ holds for all subgroups U and V with $U \geq V$. When every maximal subgroup of the group G is modular in G, G is called an *M (1)-group*. Schmidt [2] has characterized the M (1)-groups as follows:

THEOREM (Schmidt). *A finite group* G *is an* M (1)*-group if and only if* G *is supersolvable and for each complemented chief factor* H/K *of* G, $|\mathrm{Aut}_G (H/K)|$ *is prime* (*or* 1).

In this note two additional characterizations are given for finite M (1)-groups. It is shown that a finite group G is an M (1)-group whenever the following modular law holds:

$$M_1 \cap \langle V, M_2 \rangle = \langle V, M_1 \cap M_2 \rangle \tag{1}$$

for M_1 and M_2 maximal subgroups of G with $M_1 \geq V$.

Additionally it is shown that the finite group G is an M (1)-group if and only if

$$G/\Phi(G) \cong \oplus \sum_{i=1}^{n} H_i ,$$

where each H_i is a group whose order is the product of two distinct primes.

Throughout this work G denotes a finite group. $\Phi(G)$ denotes the Frattini subgroup of G, $\mathrm{Core}_G(H)$ denotes the largest normal subgroup of G contained in the subgroup H. For M a maximal subgroup of G we write $M <\cdot\, G$, while $M \leq \cdot\, G$ denotes that M is a maximal subgroup of G or $M = G$. All other notation used is standard.

In proving the results mentioned above it is convenient to use an alternate criterion for a maximal subgroup to be modular in G. This is provided by the following proposition whose proof is elementary and has been omitted.

(*) Nella seduta dell'11 giugno 1975.

PROPOSITION 1. *The maximal subgroup* M *and the subgroup* U *of the finite group* G *satisfy the modular law,*

$$U \cap \langle V , M \rangle = \langle V , U \cap M \rangle \qquad \textit{for} \quad U \geq V,$$

if and only if $U \cap M \leq \cdot U$.

As a result of Proposition 1 the modular law (1) is precisely equivalent to the condition:

If M_1 and M_2 are maximal subgroups of G then

$$(2) \qquad M_1 \cap M_2 \leq \cdot M_1 (M_2) .$$

We are now prepared to prove the first result.

THEOREM 2. *The following statements are equivalent*:

1) *If* M_1 *and* M_2 *are maximal subgroups of* G, *then*

$$M_1 \cap \langle V, M_2 \rangle = \langle V, M_1 \cap M_2 \rangle \qquad \textit{for} \quad M_1 \geq V.$$

2) *If* M_1 *and* M_2 *are maximal subgroup of* G *then*

$$M_1 \cap M_2 \leq \cdot M_1 (M_2) .$$

3) G *is an* M (1)-*group.*

Proof. As we have seen (1) and (2) are equivalent. As it is evident that (3) implies (1), it needs only to be shown that (2) implies (3). For this we let G be the minimal counter example. Two cases are now distinguished.

Case 1. G *is solvable.* Using the characterization of Schmidt for M (1)-groups it suffices to show that for H/K a complemented chief factor of G, $| H : K | = p$ and $| \mathrm{Aut}_G (H/K) | = q$ (or 1) with p and q primes. Let H/K be a complemented p-chief factor of G complemented by the maximal subgroup M. As G is the minimal counter example we may assume that $M > \mathrm{Core}_G (M) = \{1\}$; so that $K = \{1\}$, $G = MH$, $M \cap H = \{1\}$ and $C_G (H) = H$. Let $1 \neq x \in H$. As G satisfies (2), $M^x \cap M \leq \cdot M$.

For $y \in M \cap M^x$, $y = z^x$ where $z \in M$ so that $z^x z^{-1} \in M$. Since $H \lhd G$, $z^x z^{-1} \in H$ and it follows that $z^x z^{-1} = 1$. Hence x centralizes $z = y$ and $C_G (x) \geq M \cap M^x$.

As $M \cap M^x \leq \cdot M$, $(M \cap M^x) H \leq \cdot G$. G satisfies (2) so that

$$M \cap (M \cap M^x) H = M \cap M^x < \cdot (M \cap M^x) H .$$

However, since

$$C_G (x) \geq M \cap M^x, H (M \cap M^x) \geq \langle x \rangle (M \cap M^x) > M \cap M^x.$$

Therefore $\langle x \rangle = \mathrm{H}$ and $|\mathrm{H}| = p$, p a prime. As $\mathrm{C_G}(x) = \mathrm{C_G}(\mathrm{H}) = \mathrm{H}$, it follows that $\mathrm{M} \cap \mathrm{M}^x = \{1\}$ hence $|\mathrm{M}| = q$, q a prime. By the theorem of Schmidt G is an M(1)-group contrary to the choice of G.

Case II. G *is nonsolvable*. A contradiction will be arrived at by showing that each maximal subgroup of G has prime index, so that G would not only be solvable but supersolvable as well. Let M be a maximal subgroup of G and $\mathrm{N} = \langle t : t^2 = 1 \rangle$. N is a normal subgroup of G and $|\mathrm{N}| > 1$ since G is not solvable. If $\mathrm{N} \leq \mathrm{M}$, then by induction G/N is supersolvable so that $|\mathrm{G} : \mathrm{M}|$ is prime. We may therefore assume that M is a non-normal subgroup of G and that $\mathrm{G} = \langle t, \mathrm{M} \rangle$ for some involution t of G.

As G satisfies (2), $\mathrm{M} \cap \mathrm{M}^t < \cdot \mathrm{M}$; furthermore $(\mathrm{M} \cap \mathrm{M}^t)^t = \mathrm{M} \cap \mathrm{M}^t$ so that $t \in \mathrm{N_G}(\mathrm{M} \cap \mathrm{M}^t)$. Let $\mathrm{H} = \langle t \rangle (\mathrm{M} \cap \mathrm{M}^t)$, observe that $\mathrm{M} \cap \mathrm{M}^t < \cdot \mathrm{H}$ and $\mathrm{M} \cap \mathrm{M}^t \lhd \mathrm{H}$.

H is a maximal subgroup of G. For if $\mathrm{G} \cdot > \mathrm{M}_1 \geq \mathrm{H}$, then $\mathrm{M} \cdot > \mathrm{M} \cap \mathrm{M}_1 \geq \geq \mathrm{M} \cap \mathrm{M}^t$. As $\mathrm{M} \cap \mathrm{M}^t < \cdot \mathrm{M}$, $\mathrm{M} \cap \mathrm{M}^t = \mathrm{M} \cap \mathrm{M}_1$. Conversely $\mathrm{M} \cap \mathrm{M}^t = = \mathrm{M} \cap \mathrm{M}_1 < \cdot \mathrm{M}_1$. Since $\mathrm{H} > \mathrm{M} \cap \mathrm{M}_1$, $\mathrm{H} = \mathrm{M}_1$ and $\mathrm{H} < \cdot \mathrm{G}$.

Let $m \in \mathrm{M} \sim (\mathrm{M} \cap \mathrm{M}^t)$. $\mathrm{H}^m = \langle t^m \rangle (\mathrm{M} \cap \mathrm{M}^{tm})$; observe that $\mathrm{M} \cap \mathrm{M}^{tm} < \cdot \mathrm{H}^m$ and $\mathrm{M} \cap \mathrm{M}^{tm} \lhd \mathrm{H}^m$. Since $\mathrm{H} \cap \mathrm{H}^m \leq \mathrm{N_G}(\mathrm{M} \cap \mathrm{M}^t)$ and $\mathrm{H} \cap \mathrm{H}^m \leq \leq \mathrm{N_G}(\mathrm{M} \cap \mathrm{M}^{tm})$, it follows that $\mathrm{H} \cap \mathrm{H}^m \leq \mathrm{N_G}(\langle \mathrm{M} \cap \mathrm{M}^t, \mathrm{M} \cap \mathrm{M}^{tm} \rangle)$. As $\mathrm{M} \cap \mathrm{M}^t$ and $\mathrm{M} \cap \mathrm{M}^{tm}$ are both maximal subgroups of M, either $\mathrm{M} \cap \mathrm{M}^t = = \mathrm{M} \cap \mathrm{M}^{tm}$ or $\mathrm{M} = \langle \mathrm{M} \cap \mathrm{M}^t, \mathrm{M} \cap \mathrm{M}^{tm} \rangle$.

Suppose $\mathrm{M} \cap \mathrm{M}^t = \mathrm{M} \cap \mathrm{M}^{tm}$. In this case $m, t \in \mathrm{N_G}(\mathrm{M} \cap \mathrm{M}^t)$. As $\mathrm{G} = \langle t, m, \mathrm{M} \cap \mathrm{M}^t \rangle$, it follows that $\mathrm{M} \cap \mathrm{M}^t \lhd \mathrm{G}$. Were $\mathrm{M} \cap \mathrm{M}^t = \{1\}$, then H would be a maximal subgroup of G of order 2; thus would G be solvable, contrary to its choice. Were $\mathrm{M} \cap \mathrm{M}^t \neq \{1\}$, then applying induction, $\mathrm{G}/\mathrm{M} \cap \mathrm{M}^t$ is supersolvable so that $|\mathrm{G} : \mathrm{M}|$ is prime. We may therefore assume that $\mathrm{M} \cap \mathrm{M}^t \neq \mathrm{M} \cap \mathrm{M}^{tm}$.

If $\mathrm{M} = \langle \mathrm{M} \cap \mathrm{M}^t, \mathrm{M} \cap \mathrm{M}^{tm} \rangle$, then $\mathrm{H} \cap \mathrm{H}^m \leq \mathrm{N_G}(\mathrm{M}) = \mathrm{M}$. Hence $\mathrm{M} \geq \langle \mathrm{H} \cap \mathrm{H}^m, \mathrm{M} \cap \mathrm{M}^t \rangle$. As $\mathrm{H} \cap \mathrm{H}^m$ and $\mathrm{M} \cap \mathrm{M}^t$ are maximal subgroups of H it follows, since $\mathrm{H} \neq \mathrm{M}$, that $\mathrm{H} \cap \mathrm{H}^m = \mathrm{M} \cap \mathrm{M}^t$. But $\mathrm{H}^m \geq \langle \mathrm{H} \cap \mathrm{H}^m, \mathrm{M} \cap \mathrm{M}^{tm} \rangle = \langle \mathrm{M} \cap \mathrm{M}^t, \mathrm{M} \cap \mathrm{M}^{tm} \rangle = \mathrm{M}$. As $m \in \mathrm{M}$, $\mathrm{H} = \mathrm{M}$ contrary to the choice of t. Therefore $|\mathrm{G} : \mathrm{M}|$ is prime and G is supersolvable. As G had to be nonsolvable it follows that (2) implies (3).

It is clear from the work of Schmidt that a group G is an M(1)-group if and only if $\mathrm{G}/\Phi(\mathrm{G})$ is an M(1)-group. With this in mind the M(1)-groups with trivial Frattini subgroup are now investigated.

PROPOSITION 3. *Let* G *be an* M(1)*-group with trivial Frattini subgroup and* H *a subgroup of* G. *Then,*

1) $\Phi(\mathrm{H}) = \{1\}$,

2) H *is complemented, and*

3) H *is an* M(1)*-group.*

Proof. From Proposition 1 it is seen that G is an M (1)-group if and only if $M \cap U \leq \cdot U$ for all subgroups U and maximal subgroups M of G. Hence $\{1\} = H \cap \Phi(G) = \cap \{H \cap M : M < \cdot G\} \geq \cap \{M^*: M^* \leq \cdot H\} = \Phi(H)$, so that $\Phi(H) = \{1\}$. In particular if P is a Sylow p-subgroup of G, $\Phi(P) = \{1\}$ and P is elementary abelian. Hence as M (1)-groups are supersolvable, it follows from a result of P. Hall [1, Theorem 2] that every subgroup of G has a complement.

Let $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright G_3 \triangleright \cdots \triangleright G_n = \{1\}$ be a chief series of G. As G is complemented each chief factor is complemented. Hence by Schmidt's theorem $|G_i : G_{i+1}|$ is prime and $|\mathrm{Aut}_G(G_i/G_{i+1})|$ is prime (or 1) so that $C_G(G_i/G_{i-1}) \leq \cdot G$.

Let $H_i = H \cap G_i$. Since G is supersolvable,

$$H = H_0 \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq H_3 \trianglerighteq \cdots \trianglerighteq H_n = \{1\}$$

is reducible to a chief series of H by eliminating trivial factors. If H_i/H_{i+1} is a nontrivial factor then, since $|G_i : G_{i+1}| = p$ for some prime p, $G_i = H_i G_{i+1}$. Hence $C_H(H_i/H_{i+1}) = H \cap C_G(G_i/G_{i+1})$. As $C_G(G_i/G_{i+1}) \leq \cdot G$ it follows, since G is an M (1)-group, from Proposition 1 that $C_H(H_i/H_{i+1}) \leq \cdot H$; thus $|\mathrm{Aut}_H(H_i/H_{i+1})|$ is prime or 1. Since $|H_i : H_{i+1}| = |G_i : G_{i+1}| = p$, H is an M (1)-group.

From Proposition 3 it is seen that the class of M (1)-groups with trivial Frattini subgroup is both subgroup closed and factor group closed. In that the direct sum of M (1)-groups are again M (1)-groups we see that this class is also closed to direct sums. For any group G with normal subgroups N and K, $G/N \cap K$ is isomorphic to a subgroup of $G/N \oplus G/K$. Thus the class of M (1)-groups with trivial Frattini subgroup is a subgroup closed formation.

THEOREM 4. *G is an M (1)-group if and only if* $G/\Phi(G) \cong \oplus \sum_{i=1}^{n} H_i$, *where each* H_i *is a group whose order is the product of two distinct primes.*

Proof. Without loss of generality we may assume $\Phi(G) = \{1\}$.

Suppose $G \cong \oplus \sum_{i=1}^{n} H_i$, where each H_i has order the product of two distinct primes. As each H_i is an M (1)-group with trivial Frattini subgroup so is $\oplus \sum_{i=1}^{n} H_i$. Therefore by Proposition 3, G is an M (1)-group.

Suppose G is an M (1)-group, G a minimal counter example. If G has two minimal normal subgroups H and K, then by Proposition 3, G/H and G/K are M (1)-groups with trivial Frattini subgroup. By induction $G/H \cong \oplus \sum_{i=1}^{r} H_i$ and $G/K \cong \oplus \sum_{i=1+r}^{n} H_i$, where each H_i has order the product of two distinct primes. Since H and K are minimal normal subgroups

$H \cap K = \{1\}$, so that $G \lesssim G/H \oplus G/K \lesssim \oplus \sum_{i=1}^{n} H_i$. Thus we may assume G has a unique minimal normal subgroup H. As G is an M (1)-group $|H| = p$ for some prime p. $\Phi(G) = \{1\}$ implies H is complemented so that $|\mathrm{Aut}_G(H)| = q$ (or 1) for q a prime. Since $H = C_G(H)$, $|G : H| = q$ (or 1) so that $|G| = pq^a$ where $a = 0$ or 1. Therefore G satisfies the theorem, contrary to its choice, and the Theorem is proven.

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