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V. R. Chandran

On duo-rings

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Algebra. — *On duo-rings.* Nota di V. R. CHANDRAN (*), presentata (**) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si studiano gli anelli associativi in cui ogni ideale sinistro (o destro) è un ideale bilatero (duo-rings). In particolare si danno esempi non banali di «duo-rings» non commutativi, si caratterizzano alcune notevoli classi di «duo-rings»; si prova che è sempre possibile immergere un «duo-ring» regolare in un «duo-ring» regolare con unità.

INTRODUCTION

By a ring we mean an associative ring. A duo-ring is one in which every one sided ideal is two sided. As Professor Jacobson has asked (personal correspondence) for the nontrivial examples of non-commutative duo-rings in section 1 § we give such examples. In section 2 § we describe the classes of, primitive right duo-rings. In section 3 § we give an interesting characterization of (Von-Neumann) regular rings in the class of right duo-rings. In section 4 § we prove that a regular duo-ring can be embedded in a regular duo-ring with 1. In this connection, the author wishes to express his sincere thanks to Professor M. Venkataraman and Dr. K. R. Nagarajan for their valuable comments on this paper.

§ SECTION 1 § EXAMPLES

DEFINITION 1.1. A ring R (not necessarily with 1) is a right duo-ring if every right ideal in R is a two sided ideal. Similarly one can define a left duo-ring. A ring is a duo-ring if it is both right and left duo.

PROPOSITION 1.2 [2]. *A ring* R *with* 1 *is a duo-ring if and only if* $a\mathrm{R} = \mathrm{R}a$ *for each element* a *in* R. *Or, equivalently, a ring* R *with* 1 *is a duo ring if and only if given any two elements* $a, b \in \mathrm{R}$, $\exists$ *two elements* x *and* y *such that* $ab = bx$ *and* $ab = yb$.

Example 1.3. Any division ring is a duo-ring and consequently any finite or infinite direct sum of division rings is a duo-ring.

Example 1.4. A ring R is said to be strongly regular if for each element $a \in \mathrm{R}$, $\exists x \in \mathrm{R}$ such that $a = a^2 x$. Strongly regular rings are duo-rings. Refer. [5].

(*) UGC Senior Research Fellow.
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Example 1.5. Let F be a field and σ any non-identity automorphism of F. Let F $[[x]]$ denote the set of formal power series in which the equality of two elements and the addition of two elements is defined in an obvious manner. The multiplication of two elements in F $[[x]]$ is defined as follows: For any integer $n \geq 1$, define $x^n b = b^{\sigma n} x^n$ where $b \in$ F and $b^{\sigma n}$ denotes the effect on b by the automorphism σ applied n times in succession. Then this multiplication is extended by linearity of addition to all elements of F $[[x]]$. Under these operations F $[[x]]$ is a ring with identity element. Using an equivalent formulation of Proposition 1, one can actually verify (though this involves calculations) that F $[[x]]$ is a duo-ring with 1.

Example 1.6. E. H. Feller [2] has remarked that any finite dimensional Grassmann algebra over a field of characteristic p is a duo-ring with 1.

Example 1.7. One can easily check that any anti-commutative ring is a duo-ring. (A ring R is said to be anticommutative if for any two elements $a , b \in$ R, $ab = - ba$ holds).

Next we give examples of right duo-rings which are not left duo.

Example 1.8. Let F be a field with p-elements, p a prime number. We define addition termwise in the cartesian product R $=$ F $\times$ F. We define multiplication in R as follows, $(a , b) \circ (c , d) = (ac , ad)$ where $a , b , c , d \in$ F. Then one can check R is right duo and not left duo.

One also notes that the opposite ring of R is left duo and not right duo in the above example.

Example 1.9. Let F be a field of characteristic zero. Let σ be an endomorphism of F (Specifically take σ not to be an automorphism). Then one can check that R $=$ F $([[x]] , \sigma)$ with the ring operations as defined in Example 1.5 is a left duo-ring and not a right duo-ring.

§ SECTION 2 §

For definitions refer. [4].

THEOREM 2.1. *A right primitive, right duo-ring* R *is a division ring.*

Proof. Since R is right primitive, refer. [4], $(0) = (\mathrm{P} : \mathrm{R})$ where P is a modular maximal right ideal in R. Let e be a left identity modulo P. We know that $(\mathrm{P} : \mathrm{R})$ is the largest two sided ideal contained in P. Since R is right duo, P is a two sided ideal and hence $(0) = (\mathrm{P} : \mathrm{R}) = \mathrm{P}$. Now the modularity of P $(\mathrm{P} = (0))$ implies R has a left identity. Since P $(\mathrm{P} = (0))$ is a maximal right ideal in R we deduce that R is a division ring.

Remark 2.2. It is not known whether a right primitive, left duo-ring is a division ring or not.

As in Jacobson [4], using the above theorem one can deduce the following:

COROLLARY. *Any (Jacobson) semisimple right duo-ring is a dense subdirect sum of skewfields.*

§ SECTION 3 §

For definitions refer. [6].

DEFINITION 3.1. R a given ring. I an ideal in R. Then we define $\mathrm{I} = \{x/x^n \in \mathrm{I}\}$ for some positive integer n.

DEFINITION 3.2. For an ideal I in a ring R, $r(\mathrm{I}) = \{x/x \in \mathrm{R}$ and every m-system which contains x contains an element of $\mathrm{I}\}$.

Remark [6]. $r(\mathrm{I})$ is an ideal and is equal to the intersection of all prime ideals containing I.

LEMMA 3.3. *In a right duo-ring* $\sqrt{\mathrm{I}} = r(\mathrm{I})$ *for any ideal* I *in* R.

Proof. First, we show $r(\mathrm{I}) \subseteq \sqrt{\mathrm{I}}$. For let $x \in r(\mathrm{I})$. Now, the set $\{x, x^2, \cdots\}$ is clearly an m-system and thus this set contains an element of I which clearly implies $x \in \sqrt{\mathrm{I}}$. To prove $\sqrt{\mathrm{I}} \subseteq r(\mathrm{I})$, let $x \in \sqrt{\mathrm{I}}$. Then $x^n \in \mathrm{I}$ for some positive integer n. Let M be any m-system containing x. Now for the elements $x, x \in \mathrm{M}$, $\exists$ an element y_1 in R such that $xy_1x \in \mathrm{M}$. Now for the elements xy_1x and x in M, $\exists$ an element y_2 in R such that $xy_1xy_2x \in \mathrm{M}$. Thus proceeding we get elements $y_1, y_2, \cdots, y_n$ in R such that $xy_1xy_2x, \cdots, y_nx \in \mathrm{M}$. Now by induction hypothesis, assume that $(xy_1)(xy_2)\cdots(xy_{n-1})$ can be written as $x^{n-1}y$ for some $y \in \mathrm{R}$.
Then

$$xy_1xy_2, \cdots, xy_n = (xy_1, \cdots, xy_{n-1}) = (x^{n-1}y)(xy_n) = x^{n-1}(yxy_n) =$$

$$= x^{n-1}xz \,(\because \mathrm{R} \text{ is right duo, } \mathrm{R}x\mathrm{R} \subseteq \mathrm{XR} \text{ and so } yxy_n = xz) =$$

$$= x^n z \quad \text{which clearly belongs to I}.$$

Thus, M contains an element of I, namely $xy_1x, \cdots, y_nx$. Hence $x \in r(\mathrm{I})$. Thus, $r(\mathrm{I}) = \sqrt{\mathrm{I}}$.

THEOREM 3.4. *A right duo-ring* R *is regular if and only if an ideal* I *in* R *coincides with its radical* $r(\mathrm{I})$.

Proof. Assume R is regular. By the definition of $r(\mathrm{I})$ it is clear that $\mathrm{I} \subseteq r(\mathrm{I})$. To show $r(\mathrm{I}) \subseteq \mathrm{I}$, let $a \in r(\mathrm{I})$. Using Lemma 3.3, we get $a^i \in \mathrm{I}$ for some positive integer i. Since R is regular, $\exists x \in \mathrm{R}$ such that $axa = a$.

$$\text{Now,} \quad a = (ax)^i a \quad \text{using} \quad axa = a.$$

Since R is right duo, as in the proof of Lemma 3.1 one can show $(ax)^i = a^i z$ for some z in R. Now, $a = (ax)^i a = a^i z$ which clearly belongs to I.

Conversely, let $I = r(I)$ for every ideal I in R. By Lemma 3.3 we get $(o) = \sqrt{(o)} = r(o)$, i.e. R has no non-zero nilpotent elements. Now, let a be any element in R. To show $a = axa$ for some $x \in R$, let $S = \{a^2 x / x \in R\}$. Clearly, S is a right ideal of R and hence two sided. Thus by hypothesis (using Lemma 3.3) we get $S = \sqrt{S}$. Now $a^3 \in S$ which implies $a \in \sqrt{S} = S$ which in turn implies $a = a^2 x$ for some $x \in R$. Now, using the above equation one easily checks $(a - axa)^2 = 0$ which implies $a = axa$, since R has no nonzero nilpotent elements. I.e. R is regular.

§ Section 4 §

Embedding a regular duo-ring in a regular duo-ring with identity element.

LEMMA 4.1 [7]. *Let* R *be a duo-ring. Then for any three elements* a, b, c *in* R, $\exists$ *elements* x *and* y *in* R *such that* $abc = bx = yb$.

LEMMA 4.2 [8]. *In a duo-ring* R, *every idempotent is central.*

THEOREM 4.3. *Let* R *be a duo-ring. Suppose* (1) $\exists$ *a commutative ring* M *with* 1 *such that* R *can be considered as an algebra over* M *and* (2) *for each element* a *in* R, $\exists$ *an idempotent* e *in* R *such that* $ea = ae = a$. *Then* R *can be embedded as an ideal in a duo-ring with identity element.*

Proof. Let $R' = (R, M) = \{(x, i)/x \in R, i \in M\}$. Define $+$ in R' in the usual manner and define the multiplication for two elements in R' as follows: $(a, i)\circ(b, j) = (ab + ib + jb, ij)$. Under these operations one can easily check R' is an associative ring with identity element and R is in fact an ideal in R' under the mapping $a \to (a, 0)$.

To show that R' is a duo-ring, it suffices to show that $(b, j)R' = R'(b, j)$ for each element (b, j) in R' where $(b, j)R' = \{(b, j)x/x \in R'\}$. Given any two elements (a, i) and (b, j) in R' we are going to show the existence of an element (z, k) in R' such that $(a, i)\circ(b, j) = (b, j)\circ(z, k)$. Now, by hypothesis, for the elements a, b in R, $\exists$ two idempotents e_1 and e_2 such that $e_1 a = ae_1 = a$ and $e_2 b = be_2 = b$. Also by Lemma 4.2, each idempotent is central in R. Call $e = e_1 + e_2 - e_1 e_2$. Clearly e is an idempotent and one verifies that $ea = ae = a$ and $eb = be = b \cdots (1)$. Now consider the elements $a + ie, b + je, e$ in R. By Lemma 4.1, $\exists$ an element $z \in R$ such that $(a + ie)(b + je)e = (b + je)z$. Call $z' = z - ie$. Thus, $(a + ie)(b + je)e = (b + je)(z' + ie)$. Expanding and using equation (1) we get $ab + ib + ja + ije = (be)z' + ib + jez' + ije$. Subtracting ije in both sides and adding ij in both sides respectively, $ab + ib + ja + ij = b(ez') + ib + jez' + ij$,

$$\text{That is,} \qquad (a, i)(b, j) = (b, j)(ez', i)$$

$$\text{Thus,} \qquad R'(b, j) \subseteq (b, j)R' \cdots (2).$$

In a similar manner one can prove that (using Lemma 4.1) like for the elements e , $a + ie$, $b + je$, $\exists$ an element y in R such that

$$e\,(a + ie)\,(b + je) = y\,(b + je)$$

and therefore

$$(b\,,j)\,\mathrm{R}' \subseteq \mathrm{R}'\,(b\,,j)\,.$$

Thus, R' is a duo-ring.

COROLLARY 4.4. *Let* R *be a regular duo-ring. Then* R *be can be embedded as an ideal in a regular duo-ring with* 1.

Proof. Let R be regular. By Fuchs [3], $\exists$ a commutative ring M with identity element such that $\mathrm{R}' = (\mathrm{R}\,,\mathrm{M})$ (where the $+$ and $\cdot$ are defined in R' as in Theorem 4.3) is regular ring with 1. Further R is algebra over M and R is an ideal in R'. Also for each element a , $a = (ax)\,a$ and the idempotent ax is central by Lemma 4.2. Thus, as the conditions of Theorem 4.3 are satisfied $\mathrm{R}' = (\mathrm{R}\,,\mathrm{M})$ is a duo-ring.

COROLLARY 4.5. *Let* R *be a duo-ring such that for each element a in* R, $\exists$ *an idempotent e depending on a such that* $ea = ae = a$. *Then* R *can be embedded in a duo-ring with* 1.

Proof. The ring R can be considered as an algebra over the ring of integers Z. Thus the conditions of Theorem 4.3 are satisfied. Thus, $\mathrm{R}' = (\mathrm{R}\,,\mathrm{Z})$ where the binary operations are defined as in Theorem 4.3 is a duo-ring with 1.

(P.T.O).

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