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Structure theorems for a Banach Lie loop

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RENDICONTI

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SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *Structure theorems for a Banach Lie loop.* Nota di AUREL BEJANCU, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Usando la teoria delle forme differenziali su d'uno spazio di Banach, si ottengono dei teoremi di Lie per un cappio di Lie Banach G. Se G è compatto si costruisce poi un diffeomorfismo della varietà di tutte le funzioni C^r da G in G.

In the first part of this Note we obtain Lie's theorems for a Banach Lie loop, using the theory of differential forms on a Banach manifold. Let G be a compact Lie loop and $C^r(G;G)$ be the manifold of all morphisms from G to G. In the second part of this Note we construct a diffeomorphism of $C^r(G;G)$ on $C_e^r(G;G)\times G$ where $C_e^r(G;G)=\{f\in C^r(G;G);f(e)=e\}$.

1. By a *loop* we mean a set G with an operation $\pi: G\times G\to G$ satisfying the following conditions:

1) there exists a unique element $e\in G$ such that $\pi(x,e)=\pi(e,x)=x$ for every $x\in G$,

2) the equations $\pi(a,x)=b$ and $\pi(y,a)=b$ have the unique solutions $x=\psi(a,b)$ and $y=\varphi(b,a)$ respectively.

Thus the maps ψ and φ are two other operations on G.

Let **E** be a real Banach space.

DEFINITION 1.1. A Banach Lie loop modelled on **E** is a loop which has a structure of a Banach manifold modelled on **E**, of class C^∞ such that the maps π, ψ and φ are differentiable of class C^∞.

(*) Nella seduta dell'11 giugno 1975.

For a Banach Lie loop G we use the following notations:

$\mathscr{X}(G)$	– the module of all vector fields of class C^∞ on the manifold G,
$C(G; T_e G)$	– the module of all maps of class C^∞ from G to $T_e G$,
$L(T_x G; T_e G)$	– the Banach space of all linear continuous operators from $T_x G$ to $T_e G$,
$\text{Lis}(T_x G; T_e G)$	– the Banach space of all linear continuous and invertible operators from $T_x G$ to $T_e G$,
$L(TG; T_e G)$	– the Banach vector bundle over G with fibre $L(T_x G; T_e G)$ at x,
$L_x^\pi, L_x^\psi, L_x^\varphi$	– the left translations on G induced by $x \in G$ and the maps π, ψ, φ.

We studied Banach Lie groups using the differential forms which take values in a Banach space ([1]). The same method will be used in this Note in order to obtain the first two Lie theorems for Banach Lie loops.

PROPOSITION 1.1. *If* G *is a Banach Lie loop, then the vector bundle* $L(TG; T_e G)$ *has a global section* ω *differentiable of class* C^∞, *such that* $\omega(x) \in \text{Lis}(T_x G; T_e G)$ *for every* $x \in G$.

Proof. We define ω as the map

$$\omega(x) = T_x L_x^\psi \qquad \forall x \in G. \tag{1.1}$$

It is easy to check that ω is a C^∞ global section for the vector bundle $L(TG; T_e G)$. Furthermore, if we define ω^{-1} as the map

$$\omega^{-1}(x) = T_e L_x^\pi, \qquad \forall x \in G, \tag{1.2}$$

we obtain $\omega^{-1}(x) \circ \omega(x) = 1_{T_x G}$ and $\omega(x) \circ \omega^{-1}(x) = 1_{T_e G}$.

PROPOSITION 1.2. *Let* G *be a Banach Lie loop. Then the modules* $\mathscr{X}(G)$ *and* $C(G; T_e G)$ *are isomorphic.*

We use the section ω defined in Proposition 1.1 and get the maps

$$\omega : \mathscr{X}(G) \to C(G; T_e G), \langle \omega, X \rangle (x) = \omega(x)(X(x)) \qquad x \in G, X \in \mathscr{X}(G), \tag{1.3}$$

$$\omega^{-1} : C(G; T_e G) \to \mathscr{X}(G), \omega^{-1}(f)(x) = \omega^{-1}(x)(f(x)) \qquad x \in G, f \in C(G; T_e G). \tag{1.4}$$

The proof of Proposition 1.2 results from (1.3) and (1.4).

Therefore every Banach Lie loop is a parallelisable manifold. The differential form defined by (1.1) and (1.3) will play a particular role in the local theory of Banach Lie loops.

Now it is useful to introduce the map

$$A(y, x) = \omega^{-1}(x) \circ \omega(\pi(y, x)) \circ T_x L_a^{\pi}, \qquad a, x \in G. \tag{1.5}$$

This map is differentiable of class C^{∞} and verifies

$$A(e, x) = 1_{T_x G} \qquad \forall x \in G. \tag{1.6}$$

Let M, N, P be three Banach manifolds of class C^{∞} and $f: M \times N \to P$ be a C^{∞} differentiable map. Let $a \in M$ and $b \in N$. The linear tangent map to the map $x \to f(x, b)$ (resp. $y \to f(a, y)$) from M (resp. N) to P will be denoted by $T^1_{(a,b)} f$ (resp. $T^2_{(a,b)} f$). Using a local calculus we get

$$T^1_{(\pi(a,x),x)} \varphi \, T^2_{(a,x)} \pi + T^2_{(\pi(a,x),x)} \varphi = 0. \tag{1.7}$$

From (1.5) and (1.7) we obtain:

$$\begin{aligned} A(y, x) &= -T_e L_x^{\pi} \circ T_{\pi(y,x)} L^{\psi}_{\pi(y,x)} \circ (T^1_{(\pi(y,x),x)} \varphi)^{-1} \circ T^2_{(\pi(y,x),x)} \varphi = \\ &= B(\pi(y, x), x), \end{aligned} \tag{1.8}$$

$$B(x, x) = 1_{T_x G} \qquad \forall x \in G. \tag{1.9}$$

In this paragraph we are going to study local problems for Banach Lie loops so it is useful to find local expressions for the maps ω, A and B. By f_{10} (resp. f_{01}) we denote the partial Frechet differential of $f: M \times N \to P$ in the first variable (resp. in the second variable) considered in suitable local charts. Using this notation, the maps A and B have the following local expressions:

$$\omega(x) = \psi_{01}(x, x), \; A(y, x) =$$

$$= -\pi_{01}(x, 0) \circ \psi_{01}(\pi(y, x), \pi(y, x)) \circ [\varphi_{10}(\pi(y, x), x)]^{-1} \circ \varphi_{01}(\pi(y, x), x) =$$

$$= B(\pi(y, x), x).$$

DEFINITION 1.2. A local Banach Lie loop is a Banach manifold G of class C^{∞} with partial defined operation π, such that the following conditions are satisfied:

(i) there exists a unique element $e \in G$ such that $\pi(x, e) = \pi(e, x) = x$,

(ii) the equations $\pi(a, x) = b$ and $\pi(y, a) = b$ have the unique solutions $x = \psi(a, b)$ and $y = \varphi(b, a)$ respectively,

(iii) there exists a neighbourhood U of e such that the maps π, ψ and φ are differentiable of class C^{∞} on U.

Now, let U be an open set of the Banach space $\mathbf{E}$ such that $0 \in U$ and $\omega : U \to GL(\mathbf{E})$, $B : U \times U \to GL(\mathbf{E})$ be differentiable maps of class C^∞.

THEOREM 1.1. *The pair of maps* (ω, B) *defines a structure of a local Banach Lie loop on* U, *if, and only if, the differential equation*

$$\frac{dy}{dx} = \omega^{-1}(y) \circ \omega(x) \circ B(y, x) \tag{1.10}$$

is completely integrable and B *satisfies*

$$B(x, x) = 1_E \qquad \forall x \in U . \tag{1.11}$$

Proof. Suppose U is a local Banach Lie loop. Let $(x_0, y_0) \in U \times U$ and $a = \varphi(x_0, y_0)$. The map $y(x) = L_a^\pi(x)$ is a solution for the equation (1.10) and $y(x_0) = \pi(a, x_0) = \pi(\varphi(y_0, x_0), x_0) = y_0$. The condition (1.11) is the local expression of (1.9).

Conversely, if the equation (1.10) is completely integrable, let us denote by L_x its solution with initial conditions $(0, x) \in U \times U$. Define the partial operation by $\pi(x, y) = L_x(y)$, and it is obvious that we have $\pi(x, 0) = x$ for every $x \in U$. From (1.11) we obtain that the identity on U is a solution for equation (1.10) and hence $\pi(0, x) = x$ for every $x \in U$. If we consider z as a variable point U it is well known ([2]) that the map $y = \pi(z, x)$ is differentiable of class C^∞ and

$$\left(\frac{\partial \pi}{\partial z}\right)(z, x) \in GL(\mathbf{E}) .$$

Using the implicit functions' theorem we get $z = \varphi(y, x)$ and φ is differentiable of class C^∞. But all the solutions of the equation (1.10) are diffeomorphisms, so there exists a C^∞ map ψ such that $x = \psi(y, z)$. This completes the proof.

Let $\mathbf{E}, \mathbf{F}$ be two Banach spaces, $S, T \in GL(\mathbf{E})$, U be an open set of $\mathbf{E}$ and $\omega_p : U \to L_a^p(\mathbf{E}, \mathbf{F})$ be a Banach differential form of class C^∞ and of order p.

DEFINITION 1.3. The S-T-exterior differential $d_{S,T}\,\omega_p$ of the differential form ω_p is the following map

$$d_{S,T}\,\omega_p : U \to L_a^{p+1}(\mathbf{E}, \mathbf{F}) ,$$

$$(d_{S,T}\,\omega_p)(x ; u_0, \cdots, u_p) = \sum_{i=0}^{p} (-1)^i (\omega_p'(x) T u_i)(S u_0, \cdots, \widehat{S u_i}, \cdots, S u_p). \tag{1.12}$$

If in particular $T = S = 1_E$ we obtain the well known exterior differential operator for Banach differential forms ([2]). We must make the remark that $d_{S,T}$ has not the properties of the ordinary exterior differential, but it is useful to get integrability condition for equation (1.10).

We denote by $\omega_p^{1,a}$ (resp. $\omega_p^{2,a}$) the differential form of order p defined by the map $x \in U \to \omega_p(a, x) \in L_a^p(\mathbf{E}, \mathbf{F})$ (resp $x \to \omega_p(x, a)$) where

$\omega_p : U \times U \to L_a^p(\mathbf{E}, \mathbf{F})$ and $a \in U$. The right-hand side of equation (1.10) will be denoted by $f(x, y)$.

THEOREM 1.2. *Let* $\omega : U \to GL(\mathbf{E})$ *be a differential form of class* C^∞ *and* $B : U \times U \to GL(\mathbf{E})$ *be a* C^∞ *map. Then* U *is a local Banach Lie loop, if, and only if, the following condition is satisfied*:

$$(1.13)\qquad (d_{B(y,x),1}\,\omega)(x) + \omega(x)\circ[(dB^{1,y})(x) + (d_{1,f(x,y)}\,B^{2,x})(y)] + \\ + \omega(y)\circ[d_{\omega(y)\circ f(x,y),f(x,y)}\,\omega^{-1}](y) = 0\,,$$

for every $(x, y) \in U \times U$.

Proof. The integrability condition of equation (1.10) is satisfied if and only if the function

$$F(x, y, u, v) = \omega^{-1}\Big(y ; \Big(\frac{\partial\omega}{\partial x}\Big)(x ; B(y, x ; u))(v)\Big) + \\ + \omega^{-1}\Big(y ; \omega\Big(x ; \Big(\frac{\partial\beta}{\partial x}\Big)(y, x ; u)(v)\Big)\Big) + \\ + \Big(\frac{\partial\omega^{-1}}{\partial y}\Big)(y ; \omega(x ; B(y, x ; u)))(f(x, y)\cdot v) + \\ + \omega^{-1}\Big(y ; \omega\Big(x ; \Big(\frac{\partial\beta}{\partial y}\Big)(y, x ; u)(f(x, y)\cdot v)\Big)\Big)$$

is symmetric in the last two variables. Using (1.12) and the exterior differential operator for Banach differential forms $B^{1,y}$ and $B^{2,x}$ we get (1.13).

Let U be a local Banach Lie loop. We define the map $C : U \times \mathbf{E} \times \mathbf{E} \to \mathbf{E}$; $C(x ; u, v) = d\omega(x)(\omega^{-1}(x)(u) \wedge \omega^{-1}(x)(v))$. From Theorem 1.1 and Theorem 2 ([1]) we deduce that the local Banach Lie loop U is a Banach Lie group, if and only if $B(y, x) = 1_{\mathbf{E}}$ for every $(y, x) \in U \times U$. In this particular case the condition (1.13) becomes

$$d\omega(x) + \omega(y)\circ[d_{\omega(x),\omega^{-1}(y)\circ\omega(x)}\,\omega^{-1})(y) = 0\,.$$

2. In this paragraph we suppose that G is a compact Lie loop without boundary. The set of all C^r functions from G to G will be denoted by $C^r(G ; G)$ $(0 \le r < \infty)$. It is well known that $C^r(G ; G)$ is a C^s manifold $(0 \le s \le \infty)$ ([3], [4]). The operation on G will be denoted by π.

THEOREM 2.1. *The manifold* $C^r(G : G)$ *is a Lie loop of class* C^∞, *which is always infinite dimensional.*

Proof. The following mappings

$$D : C^r(G : G) \times C^r(G : G) \to C^r(G, G \times G)\,,\ D(f, g)(x) = (f(x), g(x))\,,$$
$$\pi_* : C^r(G, G \times G) \to C^r(G ; G)\quad,\quad \pi_*(h) = \pi\circ h$$

are differentiable of class C^∞. Thus the operation $\pi_*\circ D$ on $C^r(G ; G)$ is differentiable of class C^∞ and so are $\psi_*\circ D$ and $\varphi_*\circ D$. If e is the unit element of G, then the constant map $e_*(x) = e\,(x \in G)$ is the unit element of $C^r(G ; G)$.

The Banach Lie algebra $C^r(G; g)$ (g is the Lie algebra of G) is the Lie algebra of $C^r(G; G)$ and it is isomorphic to the tangent space of $C^r(G; G)$ at e_*. The differential form constructed in Paragraph 1 is an isomorphism of $C^r(G; g)$ on $\mathcal{X}(G)$. Thus the proof is complete.

The set $C^r_e(G; G) = \{f \in C^r(G; G) ; f(e) = e\}$ is a submanifold of the manifold $C^r(G; G)$.

THEOREM 2.2. *The manifold of all differentiable maps of class* C^r *from a compact Lie loop* G *to* G *is* C^r *diffeomorphic to the cartesian product of* G *with the manifold of all differentiable maps of class* C^r *from* G *to* G *which have the unit element of* G *as a fixed point.*

Proof. In order to prove this theorem we define the maps:

$$\alpha : C^r(G; G) \to C^r_e(G; G) \times G \quad , \quad \alpha(f) = (L^{\psi}_{f(e)} \circ f , f(e)) ,$$

$$\beta : C^r_e(G; G) \times G \to C^r(G; G) \quad , \quad \beta(g, x) = L^{\pi}_x \circ g .$$

It is easy to verify the equalities

$$\alpha \circ \beta = 1_{C^r_e(G;G)\times G} \quad , \quad \beta \circ \alpha = 1_{C^r(G;G)} .$$

The maps:

$$L^{\pi} : G \to C^r(G; G) \quad , \quad L^{\pi}(x) = L^{\pi}_x ,$$

$$Ev : C^r(G; G) \times G \to G \quad , \quad Ev(f, x) = f(x) ,$$

$$\Lambda : C^r(G; G) \times C^{2r}(G; G) \to C^r(G; G) \quad , \quad \Lambda(f, g) = g \circ f$$

are differentiable of class C^r. The map $f \to L^{\psi}_{f(e)} \circ f$ is obtained as the composition of maps

$$C^r(G; G) \to C^r(G; G) \times G \to C^r(G; G) \times C^r(G; G) \to C^r_e(G; G) ,$$

$$f \to (f, f(e)) \xrightarrow{1 \times L^{\psi}} (f, L^{\psi}_{f(e)}) \xrightarrow{\Lambda} L^{\psi}_{f(e)} \circ f .$$

Thus the map α is differentiable of class C^r. We consider β as the composition of the following maps:

$$C^r_e(G; G) \times G \to C^r_e(G; G) \times C^r(G; G) \to C^r(G; G)$$

$$(g, x) \xrightarrow{1 \times \pi} (g, L^{\pi}_x) \xrightarrow{\Lambda} L^{\pi}_x \circ g ,$$

and thus the proof of the theorem is complete.

Remark. The results of this paragraph hold true for the manifolds of maps which verify a condition of Hölder and for the Sobolev spaces.

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