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Special conformal motion in a special projective symmetric Finsler space

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Geometria differenziale. — *Special conformal motion in a special projective symmetric Finsler space.* Nota di H. D. PANDE e A. KUMAR, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — I due Autori avevano già definito i movimenti conformi e il Maher i movimenti proiettivi in uno spazio di Finsler simmetrico. Ulteriori restrizioni definiscono gli spazi di Finsler proiettivi simmetrici speciali studiati nella presente Nota.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n [1] (1) equipped with the $2n$ line elements $(x^i, \dot{x}^i)$ and with a positively homogeneous function $F(x^i, \dot{x}^i)$ of degree one in its directional argument. The fundamental metric (ensors are given by

$$g_{ij}(x, \dot{x}) \overset{\text{def.}}{=} \tfrac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \quad , \quad \dot{\partial}_i \equiv \partial/\partial \dot{x}^i \tag{1.1}$$

and

$$g^{ij}(x, \dot{x}) \, g_{jk}(x, \dot{x}) = \delta^i_k = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k . \end{cases} \tag{1.2}$$

These covariant and contravariant metric tensors are symmetric in their indices and are positively homogeneous of degree zero in $\dot{x}^i$. Let $X^i(x, \dot{x})$ be a vector field depending upon the directional and positional coordinates. The projective covariant derivative [6] of $X^i(x, \dot{x})$ is given by

$$X^i_{((k))} = \partial_k X^i - (\dot{\partial}_r X^i) \Pi^r_{mk} \dot{x}^m + X^r \Pi^i_{rk} \tag{1.3}$$

where $\Pi^i_{jk}(x, \dot{x})$ are called the coefficient of projective connection. They are symmetric in their lower indices and are homogeneous of degree zero in their directional arguments. These entities Π^i_{jk} are defined by

$$\Pi^i_{jk}(x, \dot{x}) = G^i_{jk} - \frac{1}{(n+1)} (2 \, \delta^i_{(j} G^r_{k)r} + \dot{x}^i G^r_{rkj}) \tag{1.4}$$

which satisfy the following identities:

$$a) \quad \Pi^i_{hkr} \dot{x}^r = 0 , \qquad b) \quad \Pi^i_{hkr} = \dot{\partial}_r \Pi^i_{hk} . \tag{1.5}$$

The commutation formula [6] for the projective covariant derivative of a tensor field $T^i_j(x, \dot{x})$ is expressed by

$$2 \, T^i_{j[((h))((k))]} = T^r_j Q^i_{hkr} - T^i_r Q^r_{hkj} - (\dot{\partial}_r T^i_j) Q^r_{hk} \tag{1.6}$$

(*) Nella seduta dell'8 marzo 1975.

(1) Numbers in brackets refer to the References given at the end of the paper.

where

(1.7) $$Q^i_{jkh}(x,\dot x) \overset{\text{def.}}{=} 2\,\{\dot\partial_{[h}\,\Pi^i_{k]j} - (\dot\partial_r\,\Pi^i_{j[k})\,\Pi^r_{h]s}\,\dot x^s + \Pi^r_{j[k}\,\Pi^i_{h]r}\}\ ^{(2)}.$$

The projective entities $Q^i_{jkh}(x,\dot x)$ satisfy the following identities [6]

(1.8) $$a)\ \ Q^i_{jhk}\,\dot x^j = Q^i_{hk}\,, \qquad b)\ \ Q^i_{hk}\,\dot x^h = Q^i_k$$
$$c)\ \ Q^i_{jki} = Q_{jk}\,, \qquad d)\ \ Q^i_{ki} = Q_k\,.$$

Let us consider an infinitesimal point transformation

(1.9) $$\bar x^i = x^i + v^i(x)\,\mathrm{d}t$$

where $v^i(x)$ is any vector field and $\mathrm{d}t$ is an infinitesimal constant. The Lie derivative of a tensor field T^i_j and the projective connection coefficient is given by

(1.10) $$\mathcal{L}_v\,T^i_j = T^i_{j((h))}\,v^h - T^h_j\,v^i_{((h))} + T^i_h\,v^h_{((j))} + (\dot\partial_h\,T^i_j)\,v^h_{((r))}\,\dot x^r$$

and

(1.11) $$\mathcal{L}_v\,\Pi^i_{jk} = v^i_{((j))((k))} + Q^i_{hjk}\,v^h + \Pi^i_{jkh}\,v^h_{((r))}\,\dot x^r.$$

We easily obtain the following commutation formulae:

(1.12) $$\dot\partial_k\,(\mathcal{L}_v\,T^i_j) - \mathcal{L}_v\,(\dot\partial_k\,T^i_j) = 0\,,$$

(1.13) $$\mathcal{L}_v\,T^i_{j((k))} - (\mathcal{L}_v\,T^i_j)_{((k))} = T^h_j\,\mathcal{L}_v\,\Pi^i_{kh} - T^i_h\,\mathcal{L}_v\,\Pi^h_{kj} - (\dot\partial_h\,T^i_j)\mathcal{L}_v\,\Pi^h_{ks}\,\dot x^s$$

and

(1.14) $$2\,\mathcal{L}_v\,\Pi^i_{h[k((j))]} = \mathcal{L}_v\,Q^i_{jkh} + 2\,\mathcal{L}_v\,\Pi^s_{m[j}\,\Pi^i_{k]sh}\,\dot x^m.$$

The conformal transformation is characterised by [1]

(1.15) $$\bar g^{ij}(x,\dot x) = e^{2\sigma}\,g_{ij}(x,\dot x)$$

where $\sigma = \sigma(x)$ and $\bar g_{ij}(x,\dot x)\,g_{ij}(x,\dot x)$ are the two metric tensors obtained by the two metric functions $\bar F(x,\dot x)$ and $F(x,\dot x)$ respectively. Under the conformal change we have the following entities:

(1.16) $$\bar G^i(x,\dot x) = G^i(x,\dot x) - \sigma_m\,B^{im}$$

(1.17) $$\bar G^i_h(x,\dot x) = G^i_h(x,\dot x) - \sigma_m\,B^{im}_h$$

(1.18) $$\bar G^i_{hk}(x,\dot x) = G^i_{hk}(x,\dot x) - \sigma_m\,B^{im}_{hk}$$

(1.19) $$\bar G^i_{hkr}(x,\dot x) = G^i_{hkr}(x,\dot x) - \sigma_m\,B^{im}_{hkr}\,.$$

(2) $2\,A_{[hk]} = A_{hk} - A_{kh}\,,\ 2\,A_{(hk)} = A_{hk} + A_{kh}\,.$

With the help of above equations the projective connection coefficient $\Pi^i_{hk}(x, \dot{x})$ in F_n is given by

$$\overline{\Pi}^i_{hk} = \Pi^i_{hk} - \sigma_s \left\{ B^{is}_{hk} - \frac{1}{(n+1)} (\delta^i_{(h} B^{rs}_{k)r} + \dot{x}^i B^{rs}_{rkh}) \right\} \tag{1.20}$$

where $B^{hk}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2} F^2 g^{hk} - \dot{x}^h \dot{x}^k$ is positively homogeneous of degree two in $\dot{x}^i$ and satisfies the relations

$$\text{(1.21)} \quad a) \ B^{rh}_k \dot{x}^k = 2 B^{rh}, \qquad b) \ B^{ih}_{jkm} \dot{x}^j = 0 \ , \quad B^{rm}_{kj} \dot{x}^k = B^{rm}_j$$

$$B^{ih}_k \overset{\text{def.}}{=} \dot{\partial}_k B^{ih} \ , \quad B^{ih}_{kr} \overset{\text{def.}}{=} \dot{\partial}_k B^{ih}_r \ , \quad B^{ih}_{jkr} \overset{\text{def.}}{=} \dot{\partial}_j B^{ih}_{kr} .$$

2. SPECIAL PROJECTIVE SYMMETRIC FINSLER SPACE

DEFINITION 2.1. *An* F_n *is said to be a projective symmetric* F_n (PS — F_n) *if the projective covariant derivative of* $Q^i_{jhk}(x, \dot{x})$ *vanishes identically*; i.e.

$$Q^i_{jhk((r))} = 0 . \tag{2.1}$$

The following results also hold in this space:

$$\text{(2.2)} \qquad a) \ Q^i_{jk((r))} = 0 \ , \qquad b) \ Q^i_{k((r))} = 0 .$$

If the infinitesimal point change (1.9) implies that the magnitudes of vectors defined in the same tangent space are proportional and the angles between two directions are also the same with respect to the metrics then it is called a conformal motion [4] in F_n. $\mathcal{L}_v \Pi^i_{jk}$ and $\overline{\Pi}^i_{jk}$ are the variations of $\Pi^i_{jk}(x, \dot{x})$ under infinitesimal point transformation and conformal change respectively. The two transformations will coincide if the corresponding variations are the same.

Thus we have

THEOREM 2.1. *A necessary and sufficient condition that the infinitesimal change* (1.9) *be a special conformal motion is that*

$$\mathcal{L}_v \Pi^i_{jk} \overset{\text{def.}}{=} \overline{\Pi}^i_{jk} - \Pi^i_{jk} = -\sigma_s \left\{ B^{is}_{jk} - \frac{1}{(n+1)} (2 \delta^i_{(j} B^{rs}_{k)r} + \dot{x}^i B^{rs}_{rkj} \right\} . \tag{2.3}$$

The following result also holds for the special conformal motion:

$$\mathcal{L}_v \Pi^i_{hkr} = -\sigma_s \left\{ B^{is}_{hkr} - \frac{1}{(n+1)} (\delta^i_h B^{ms}_{mkr} + \delta^i_k B^{ms}_{mhr} + \delta^i_r B^{ms}_{mhk} + \dot{x}^i B^{ms}_{mhkr}) \right\} . \tag{2.4}$$

DEFINITION 2.1. *A Finsler space* F_n *admitting* (2.1) *and* (2.3) *is called a special conformal symmetric Finsler space.*

Applying commutation formula (1.13) *to the projective entity* Q^i_j *and using equations* (2.2 *b*) *and* (2.3), *we obtain*

$$(2.5)\qquad (\mathcal{L}_v Q^i_j)_{((k))} = \sigma_s \Big[Q^h_j B^{is}_{kh} - Q^i_h B^{hs}_{kj} - (\dot\partial_h Q^i_j) B^{hs}_k - \\ - \frac{1}{(n+1)} \{Q^h_j (\delta^i_k B^{rs}_{rh} + B^i_h B^{rs}_{rk} + \dot x^i B^{rs}_{rhk}) - \\ - Q^i_h (2\,\delta^h_{(k} B^{rs}_{j)r} + \dot x^h B^{rs}_{rjk}) - (\dot\partial_h Q^i_j)(\delta^h_k B^{rs}_r + \dot x^h B^{rs}_{rk})\}\Big].$$

Contracting with respect to the indices i and j in the above equation, we get

$$(2.6)\qquad (\mathcal{L}_v Q^i_i)_{((k))} = -\sigma_s \Big[(\dot\partial_h Q^i_i)\Big\{B^{hs}_k - \frac{1}{(n+1)}(\delta^h_k B^{rs}_r + \dot x^h B^{rs}_{rk})\Big\}\Big].$$

Transvecting (2.6) with respect to $\dot x^k$ and noting (1.21), we get

$$(2.7)\qquad (\mathcal{L}_v Q^i_i)_{((k))}\,\dot x^k + 2\,\sigma_s \Big[(\dot\partial_h Q^i_i)\Big\{B^{hs} - \frac{1}{(n+1)} B^{rs}_r \dot x^h\Big\}\Big] = 0.$$

Thus we have

THEOREM 2.1. *If the infinitesimal transformation* (1.9) *defines a special conformal motion in a* PS — F_n *then equation* (2.7) *is satisfied.*

From equation (2.3) it is clear that the vanishing of the function $B^{ih}(x, \dot x)$ is a necessary and sufficient condition for the special conformal motion to become a projection affine motion. Therefore, we have

THEOREM 2.2. *In order that a special conformal motion admitted in a* PS — F_n *becomes an projective affine motion in the same space it is necessary and sufficient that* $(\mathcal{L}_v Q^i_i)_{((k))}\,\dot x_k$ *must vanish.*

The Lie-derivative of $Q^i_{jkh}(x, \dot x)$ can be obtained with the help of commutation formula (1.14) which on transvection by $\dot x^j$ and in view of equations (1.5 a) and (1.8 a) gives

$$(2.8)\qquad \mathcal{L}_v Q^i_{jk} = 2\,\mathcal{L}_v \Pi^i_{h[k((j))]}\,\dot x^h.$$

In view of (2.3) and (1.21 b), the above equation reduces to

$$(2.9)\qquad \mathcal{L}_v Q^i_{jk} = -2\Big[\sigma_s\Big\{B^{is}_{[k((j))]} - \frac{1}{(n+1)}(B^{rs}_{r[k((j))]}\dot x^i + B^{rs}_{r[((j))}\delta^i_{k]}\Big\} + \\ + \sigma_{s[((j))}\Big\{B^{is}_{k]} - \frac{1}{(n+1)}(B^{rs}_{k]r}\dot x^i + \delta^i_{k]} B^{rs}_r\Big\}\Big].$$

Multiplying (2.9) by $\dot x^j$ and contracting with respect to indices i and k, we obtain

$$(2.10)\qquad \mathcal{L}_v Q^i_i = 2\Big\{(\sigma_s B^{is})_{((i))} - \frac{1}{(n+1)}(B^{rs}_r \sigma_s)\,\dot x^i\Big\}.$$

Substituting equation (2.10) in (2.7), we obtain

$$(2.11)\qquad \Big[(\sigma_s B^{is})_{((i))((k))} - \frac{1}{(n+1)}(B^{rs}_r \sigma_s)_{((i))((k))}\,\dot x^i\Big]\dot x^k + \\ + \sigma_s\Big\{\dot\partial_h Q^i_i\Big(B^{hs} - \frac{1}{(n+1)} B^{rs}_r \dot x^h\Big)\Big\} = 0.$$

Thus we have

THEOREM 2.3. *In a* $PS - F_n$ *if the infinitesimal transformation* (1.9) *defines a special conformal motion then equation* (2.11) *holds.*

With the help of the operators $((k))$ and $\dot\partial_h$, we obtain the following commutation formula:

$$\dot\partial_h (Q^i_{jk((m))}) - (\dot\partial_h Q^i_{jk})_{((m))} = Q^s_{jk} \Pi^i_{hms} - 2 Q^i_{s[k} \Pi^s_{j]hm}, \tag{2.12}$$

which in view of equations (2.1) and (2.2 a) reduces to the following form:

$$Q^s_{jk} \Pi^i_{hms} - 2 Q^i_{s[k} \Pi^s_{j]hm} = 0. \tag{2.13}$$

Contracting (2.13) with respect to indices i and j and using (1.8 a), we get

$$Q_j \Pi^j_{khm} = 0. \tag{2.14}$$

By taking the Lie derivative of (2.14) and using equations (2.4) and (2.9), we obtain

$$\Pi^j_{khm} \left[\sigma_s \left\{B^{is}_j - \frac{1}{(n+1)} (B^{rs}_{rj} \dot x^i + \delta^i_j B^{rs}_r)\right\}\right]_{((i))} + \tag{2.15}$$

$$+ Q_j \left[\sigma_s \left\{B^{js}_{khm} - \frac{1}{(n+1)} (\delta^j_m B^{rs}_{krh} + \delta^j_h B^{rs}_{krm} + \delta^j_k B^{rs}_{hrm} + \dot x^j B^{rs}_{rkhm})\right\}\right] = 0.$$

Thus we have

THEOREM 2.4. *For a special conformal motion the equation* (2.15) *is satisfied in a* $PS - F_n$.

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