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Geometria differenziale. — *General decomposition of pseudo-projective curvature tensor field in recurrent Finsler space.* Nota di A. KUMAR e G. C. DUBEY, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Proprietà di decomposizione del campo tensoriale di curvatura pseudo-proiettiva in uno spazio di Finsler.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n [1] (1) equipped with a fundamental metric function satisfying all the required conditions imposed upon it and positively homogeneous of degree one in its directional arguments. The fundamental metric tensor $g_{ij}(x, \dot{x})$ of F_n is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}) \quad ^{(2)} \tag{1.1 a}$$

and

$$g^{ij} g_{jk} = \delta^i_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k . \end{cases} \tag{1.1 b}$$

The entities g^{ij} and g_{ij} are symmetric in their indices and are homogeneous of degree zero in $\dot{x}^i$'s. Let $X^i(x, \dot{x})$ be a contravariant vector field depending upon the directional and positional arguments. The Berwald covariant derivative of $X^i(x, \dot{x})$ with respect to x^k is given by

$$X^i_{(k)} = \partial_k X^i - (\dot{\partial}_h X^i) G^h_k + X^h G^i_{hk} , \tag{1.2}$$

where

$$G^i(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{4} g^{ih} \{ 2 \partial_{(j} g_{k)h} - \partial_h g_{jk} \} \dot{x}^j \dot{x}^k \tag{1.3}$$

are Berwald's connection coefficients which are positively homogeneous of degree two in its directional arguments.

The Berwald curvature tensor fields $H^i_{hjk}(x, \dot{x})$ resulting from the covariant derivative (1.2) are given by

$$H^i_{hjk}(x, \dot{x}) = 2 \{ \partial_{[k} G^i_{j]h} - G^i_{rh[j} G^r_{k]} + G^r_{h[j} G^i_{k]r} \} \tag{1.4}$$

(*) Nella seduta dell'11 gennaio 1975.

(1) The numbers in brackets refer to the references given in the end of the paper.

(2) $\dot{\partial}_i \equiv \partial/\dot{x}$, $\partial_i \equiv \partial/\partial^i$ and $\dot{x}^i \equiv dx^i/dt$.

(3) $2 A_{(hk)} = A_{hk} + A_{kh}$ and $2 A_{[hk]} = A_{hk} - A_{kh}$.

and satisfy the following identities:

(1.5) $$H^i_{[jkh]} = 0 \quad ^{(4)}.$$

The projective deviation tensor field $W^i_l(x, \dot{x})$ is given by

(1.6) $$W^i_l(x, \dot{x}) = H^i_l - H\delta^i_l - \dot{x}^i(\dot{\partial}_r H^r_l - \dot{\partial}_l H)/(n+1)$$

which satisfies the following Bianchi identity:

(1.7) $$W^i_{[lhj]} = 0 .$$

The pseudo-projective tensor field $W^{*i}_j(x, \dot{x})$ has been obtained in [3] as

(1.8) $$W^{*i}_j(x, \dot{x}) = aW^i_j + bH^i_j ,$$

where a and b are scalar functions of $(x, \dot{x})$ and homogeneous of degree zero in $\dot{x}^i$. Two further pseudo-projective curvature tensor fields are also obtained in [3] as

(1.9) $$W^{*i}_{hj}(x, \dot{x}) \stackrel{\text{def.}}{=} \frac{2}{3}\dot{\partial}_{[h} W^{*i}_{j]} = aW^i_{hj} + bH^i_{hj} + \frac{2}{3}[\dot{\partial}_{[h} a W^i_{j]} + \dot{\partial}_{[h} b H^i_{j]}]$$

and

(1.10) $$W^{*i}_{lhj}(x, \dot{x}) \stackrel{\text{def.}}{=} \dot{\partial}_l W^{*i}_{hj} = \frac{2}{3}\dot{\partial}^2_{l[h} W^{*i}_{j]} = aW^i_{lhj} + bH^i_{lhj} + \dot{\partial}_l a W^i_{hj} +$$
$$+ \dot{\partial}_l b H^i_{hj} + \frac{2}{3}[\dot{\partial}^2_{l[h} a W^i_{j]} + \dot{\partial}_{[h} a \dot{\partial}_{\langle l\rangle} W^i_{j]} \;^{(5)} + \dot{\partial}^2_{l[h} b H^i_{j]} + \dot{\partial}_{[h} b \dot{\partial}_{\langle l\rangle} H^i_{j]}] .$$

The following identities for pseudo-projective curvature tensor fields are also obtained in [3]:

(1.11) $$W^{*i}_{[lhj]} = 0 ,$$

(1.12) $$W^{*}_{[l\langle h\rangle j]} = \frac{1}{3}\dot{\partial}^2_{h[l} W^{*i}_{j]} ,$$

1.13) $$W^{*i}_{l[hj(s)]} = \dot{\partial}_l W^{*i}_{[hj(s)]} - W^{*r}_{[hj} G^i_{s]rl} ,$$

(1.14) $$W^{*}_{lkhj} + W^{*}_{klhj} = \frac{2}{3}[g_{ik}\dot{\partial}^2_{l[h} W^{*i}_{j]} + g_{il}\dot{\partial}^2_{k[j} W^{*i}_{h]}] ,$$

(1.15) $$W^{*}_{lkhj} + W^{*}_{jhkl} = \frac{2}{3}[g_{ik}\dot{\partial}^2_{l[h} W^{*i}_{j]} + g_{ih}\dot{\partial}^2_{j[k} W^{*i}_{l]}]$$

and

(1.16) $$W^{*}_{lkhj} - W^{*}_{ljhk} + W^{*}_{hjlk} - W^{*}_{hklj} = \frac{2}{3}[g_{i[k}\dot{\partial}^2_{j]l} W^{*i}_{h} + g_{i[k}\dot{\partial}^2_{j]h} W^{*i}_{l}] .$$

(4) $A_{[hkj]} = {}^1/_3\,[A_{hkj} + A_{kjh} + A_{jhk}]$.

(5) The indices in brackets $\langle\ \rangle$ are free from symmetric and skew symmetric parts.

DEFINITION 1.1. An n-dimensional Finsler space whose pseudo-projective curvature tensor satisfies the relation

$$W^{*i}_{ihj(s)} = u_s W^{*i}_{lhj}, \tag{1.17}$$

where

$$u_s = u_s(x) \tag{1.18}$$

is called pseudo-projective recurrent Finsler space, and $u_s(x)$ is any covariant vector independent of directional argument. For brevity from here after we shall denote a pseudo-projective recurrent Finsler space by F^*_n.

2. DECOMPOSITION OF PSEUDO-PROJECTIVE CURVATURE TENSOR FIELD W^{*i}_{lhj}

DEFINITION 2.1. Let us consider the decomposition of pseudo-projective curvature tensor field W^{*i}_{lhj} as

$$W^{*i}_{lhj} = Z^i_l \Psi_{hj} \tag{2.1}$$

and

$$Z^i_l u_i = p_l \tag{2.2}$$

where Z^i_l, Ψ_{hj} are tensor fields and u_i, p_s are recurrent vectors and decompose vector fields respectively in F^*_n.

THEOREM 2.2. *In view of the decomposition* (2.1), *the identities for pseudo-projective curvature tensor field takes the form*

$$p_{[l} \Psi_{hj]} = 0 \tag{2.3}$$

and

$$3 \Psi_{h[j} p_{l]} = u_i \dot\partial^2_{h[l} W^{*i}_{j]}. \tag{2.4}$$

Proof. In view of equation (2.1), the Bianchi identity (1.11) takes the form

$$Z^i_{[l} \Psi_{hj]} = 0. \tag{2.5}$$

Transvecting (2.5) by u_i and noting equation (2.2), we obtain the required result (2.3).

Again using equations (1.12) and (2.1), we get

$$\Psi_{h[j} Z^i_{l]} = \frac{1}{3} \dot\partial^2_{h[l} W^{*i}_{j]} \tag{2.6}$$

which on multiplying by u_i and in view of equation (2.2) yields the result (2.4).

THEOREM 2.2. *In view of the decomposition* (2.1) *the identities for pseudo-projective curvature tensor field satisfy the relation*

$$p_l u_{[s} \Psi_{hj]} = u_i [\dot\partial_l W^{*i}_{[hj(s)]} - W^{*\gamma}_{[hj} G^i_{s]\gamma l}]. \tag{2.7}$$

Proof. In view of equations (1.17) and (2.1) equation (1.13) reduces to

$$u_s \Psi_{hj} Z_l^i = \dot{\partial}_l W^{*i}_{[hj(s)]} - W^{*r}_{[hj} G^i_{s]rl} . \tag{2.8}$$

Multiplying (2.8) by u_i and noting equation (2.2), we get the theorem

THEOREM 2.3. *In view of decomposition* (2.1) *the identities for* W^*_{lkhj} *take the form*

$$Z_{lk} \Psi_{hj} + Z_{kl} \Psi_{jh} = \frac{2}{3} [g_{ik} \dot{\partial}^2_{l[h} W^{*i}_{j]} + g_{il} \dot{\partial}^2_{k[j} W^{*i}_{h]}] \tag{2.9}$$

$$Z_{lk} \Psi_{hj} + Z_{jh} \Psi_{kh} = \frac{2}{3} [g_{ik} \dot{\partial}^2_{l[h} W^{*i}_{j]} + g_{ih} \dot{\partial}^2_{j[k} W^{*i}_{l]}] \tag{2.10}$$

and

$$Z_{lk} \Psi_{hj} - Z_{lj} \Psi_{hk} + Z_{hj} \Psi_{lk} - Z_{hk} \Psi_{lj} = \tag{2.11}$$
$$= \frac{2}{3} [g_{i[k} \dot{\partial}^2_{j]l} W^{*i}_h + g_{i[k} \dot{\partial}^2_{j]h} W^{*i}_l] ,$$

where

$$a) \quad Z_{lk} \overset{\text{def.}}{=} g_{ik} Z^i_l \qquad b) \quad W^*_{lkhj} \overset{\text{def.}}{=} g_{ik} W^{*i}_{lhj} . \tag{2.12}$$

Proof. Multiplying (2.1) by g_{ik} and noting equation (2.12), we get

$$W^*_{lkhj} = Z_{lk} \Psi_{hj} . \tag{2.13}$$

In view of equation (2.13) the identities (1.14), (1.15) and (1.16) reduce to results (2.9), (2.10) and (2.11).

THEOREM 2.4. *In an n-dimensional Finsler space* F_n *if* Z^i_j *is covariantly invariant then the tensor field* Ψ_{kh} *will be a recurrent tensor field.*

Proof. Differentiating (2.1) covariantly in the sense of Berwald and using equations (1.17), we get

$$u_s W^{*i}_{lhj} = Z^i_{l(s)} \Psi_{hj} + Z^i_l \Psi_{hj(s)} . \tag{2.14}$$

If Z^i_l is covariantly constant (i.e. $Z^i_{l(m)} = 0$) then the above equation reduces to

$$u_s W^{*i}_{lhj} = Z^i_l \Psi_{hj(s)} . \tag{2.15}$$

In view of equation (2.1), the above equation yields

$$Z^i_l (\Psi_{hj(s)} - u_s \Psi_{hj}) = 0 \tag{2.16}$$

Since Z^i_l is a non zero vector, we have

$$\Psi_{hj(s)} = u_s \Psi_{hj} \tag{2.17}$$

which proves the theorem.

THEOREM 2.5. *Under the decomposition* (2.1) *the pseudo-projective curvature tensor field* W^{*i}_{lhj} *and* Ψ_{kh} *satisfy the relation*

$$p_m W^{*l}_{lhj} u_i = p_l p_m \Psi_{hj} = p_l (u_h B_{mj} - u_j B_{mh}) + \tag{2.18}$$
$$+ \frac{1}{3} u_i Z^s_m \{ \dot\partial_l W^{*i}_{[hj(s)]} - W^{*r}_{[hj} G^i_{s]rl} \} .$$

Proof. Multiplying (2.1) by $u_i p_m$, we get

$$u_i p_m W^{*i}_{lhj} = p_l p_m \Psi_{hj} . \tag{2.19}$$

Multiplying (2.7) by Z^s_m and using equation (2.2) and the skew symmetric properties of the function Ψ_{hk}, we get

$$p_l p_m \Psi_{hj} = p_l (B_{mj} u_h - u_j B_{mh}) + \frac{1}{3} u_i Z^s_m \{ \dot\partial_l W^{*i}_{[hj(s)]} - W^{*r}_{[hj} G^i_{s]rl} \}. \tag{2.20}$$

Assuming that $B_{mj} = Z^l_m \Psi_{jl}$ and noting equations (2.19) and (2.20) we get the result (2.18).

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