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Contractive mappings and periodically perturbed non-conservative systems

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Equazioni differenziali ordinarie. — *Contractive mappings and periodically perturbed non-conservative systems.* Nota di ROLF REISSIG, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Questa Nota si collega ad un'altra di J. Mawhin sui sistemi periodici conservativi perturbati.

La Nota di Mawhin si basa su un teorema di rappresentazione astratta in uno spazio di Hilbert.

L'Autore prova che con un procedimento più diretto, proprio dei problemi vibratori, si perviene ad un'estensione del risultato di Mawhin anche per sistemi non conservativi con smorzamento lineare.

Consider the differential equation

$$x'' + cx' + g(x) = e(t) \equiv e(t + 2\pi) \tag{1}$$

where c is a real constant, $g(x) \in C^1(\mathbf{R})$ and $e(t) \in C^0(\mathbf{R})$.

Following an idea of Mawhin [8] let us prove:

THEOREM 1. *Equation* (1) *admits a uniquely determined solution* $x(t) \equiv x(t + 2\pi)$ *if*

$$m^2 < q \leqq g'(x) \leqq p < (m+1)^2 \qquad \forall x \in \mathbf{R} \tag{2}$$

(*m a non-negative integer*).

The proof is based on Banach's contractive mapping principle; it is constructive since the Picard iterative method can be applied in order to get an approximate periodic solution.

Let H be the complex function space $L^2[0, 2\pi]$ supplied with the inner product

$$(x, y) = (2\pi)^{-1} \int_0^{2\pi} x(t)\, \bar{y}(t)\, \mathrm{d}t$$

and with the corresponding norm $\|x\| = (x, x)^{1/2}$. It is a well-known fact that this Hilbert space is separable; the functions $\varphi_n(t) = e^{int}$ (n integer) form a complete ON-system. Hence each function $x(t) \in H$ can be represented as a (norm convergent) Fourier series

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n \varphi_n(t) \,,\; a_n = (x, \varphi_n).$$

(*) Nella seduta del 10 maggio 1975.

Introduce the subspace D_L consisting of all functions $x(t) \in H$ which possess first and second order Lebesgue derivatives in H and which satisfy periodic boundary conditions:

$$x(0) = x(2\pi), \; x'(0) = x'(2\pi).$$

Define the linear operator $L : D_L \to H$,

$$x \mapsto Lx \equiv x'' + cx'.$$

Then differential equation (1) can be generalized as

$$Lx - Nx = y \tag{3}$$

where $x \in D_L$ and $y \in H$ (arbitrary). The nonlinear operator N: $x \mapsto -g(x)$ maps the Hilbert space H into itself since the function g is linearly bounded, $|g(x)| \leqq p|x| + |g(0)|$. According to Mawhin a real constant v, $m^2 < < \nu < (m+1)^2$, is chosen, and (3) is replaced by

$$Ax - Bx = y \tag{4}$$

where $A = L + \nu I$, $B = N + \nu I$ (I operator of identity in H).

Let us show that $A : D_L \to H$ is one-to-one, $A(D_L) = H$, and A^{-1} is completely continuous. The classical solution of equation

$$x'' + cx' + \nu x = e(t) \equiv e(t + 2\pi) \tag{5}$$

with periodic boundary conditions can be represented in the form

$$x(t) = \int_0^{2\pi} k(t-s)\, e(s)\, ds \tag{6}$$

by means of the Green function $k(t) \equiv k(t + 2\pi)$ which is continuous and piecewise smooth: $k'(t) \in C^0(0, 2\pi)$, $g'(0+) - g'(2\pi -) = 1$. Using an arbitrary $y \in H$ and introducing the functions

$$x(t) = (Ky)(t) \equiv \int_0^{2\pi} k(t-s)\, y(s)\, ds,$$

$$v(t) = (K'y)(t) \equiv \int_0^{2\pi} k'(t-s)\, y(s)\, ds,$$

$$w(t) = y(t) - \nu x(t) - c\, v(t)$$

we can verify that $x(t) \in D_L$, $v(t) \in C^0$, $w(t) \in L^2$ and

$$x(t) = x(0) + \int_0^t v(s)\, ds \; [x(0) = x(2\pi)],$$

$$v(t) = v(0) + \int_0^t w(s)\, ds \; [v(0) = v(2\pi)].$$

Note that an approximation

$$\{y_n\} \subset C^0 \subset H \ , \ \| y - y_n \| \to 0 \ (n \to \infty)$$

yields

$$(Ky_n)(t) \to (Ky)(t) \ , \ (K' y_n)(t) \to (K' y)(t)$$

as $n \to \infty$, uniformly with respect to $t \in [0, 2\pi]$. Consequently, we obtain

$$AKy = y \ \forall y \in H \ , \text{i.e. } AK : \text{identity in } H.$$

But the converse, $KAx = x \ \forall x \in D_L$, i.e. KA : identity in D_L, is valid, too: When $Au = 0$, $u \in D_L$ this function can be assumed as C^2; but the classical homogeneous boundary value problem admits the only solution $u(t) \equiv 0$.

Summarizing we have $K = A^{-1}$, a bounded linear operator defined on H:

$$\sup_t |(Ky)(t)| \leqq 2\pi \| k \| \| y \| .$$

Let λ be an eigenvalue of A^{-1}, $\varphi(t)$ a (normed) corresponding eigenfunction:

$$A^{-1} \varphi = \lambda\varphi \ (\varphi \in D_L \ , \lambda \neq 0)$$

or, equivalently:

$$\lambda(\varphi'' + c\varphi' + \nu\varphi) = \varphi \ [\varphi(0) = \varphi(2\pi) \ , \ \varphi'(0) = \varphi'(2\pi)] .$$

Then we calculate:

$$\varphi(t) = e^{int} \ (n \text{ an arbitrary integer}),$$

$$\lambda = \lambda_n = ((\nu - n^2) + inc)^{-1}.$$

The normed eigenfunctions of A^{-1} form the complete ON-system mentioned above. Therefore A^{-1} can be described in the following simple way:

$$y = \sum_n b_n \varphi_n \ , \ x = A^{-1} y = \sum_n a_n \varphi_n \qquad \text{where} \quad a_n = \lambda_n b_n .$$

As a result,

$$\| A^{-1} \| \geqq \| A^{-1} \varphi_n \| = | \lambda_n | = ((\nu - n^2)^2 + n^2 c^2)^{-1/2} \ \forall n \in \mathbf{N}$$

and

$$\| A^{-1} \| = \sup \frac{\| A^{-1} y \|}{\| y \|} \leqq \sup_{n \in \mathbf{N}} | \lambda_n | .$$

Hence

$$\text{(7)} \qquad \| A^{-1} \| = \sup_{n \in \mathbf{N}} | \lambda_n | \leqq \max ((\upsilon - m^2)^{-1}, ((m+1)^2 - \upsilon)^{-1})$$

(norm of A^{-1} in case $c = 0$).

In order to show that A^{-1} maps each bounded subset of H into a compact set consider a sequence $\{y_n\} \subset H$ $(\| y_n \| \leqq R)$ and denote $x_n = A^{-1} y_n$.

Since

$$\sup_t |x_n(t)| \leqq 2\pi \|k\| \|y_n\| \leqq X,$$

$$\sup_t |x_n'(t)| \leqq 2\pi \|k'\| \|y_n\| \leqq X'$$

the sequence $\{x_n(t)\}$ is equibounded and equicontinuous, and it contains a uniformly convergent subsequence. Clearly, the (continuous) limit function of this subsequence is also its limit in the H-norm.
Consider, once more, equation (4) which can be written as

$$x = A^{-1} Bx + A^{-1} y. \tag{8}$$

Taking account of the estimate:

$$|(Bx_1)(t) - (Bx_2)(t)| = |\nu(x_1(t) - x_2(t)) - (g(x_1(t)) - g(x_2(t)))|$$
$$\leqq \gamma |x_1(t) - x_2(t)| \quad \text{(a.e.)}$$

where $\gamma = \max(|q - \nu|, |p - \nu|)$ we conclude that

$$\|Bx_1 - Bx_2\| \leqq \gamma \|x_1 - x_2\|. \tag{9}$$

Thus, the mapping

$$u \mapsto A^{-1} Bu + A^{-1} y \tag{10}$$

of H into H is a contraction in case

$$\gamma \|A^{-1}\| < 1 \tag{11}$$

which is realized by a suitable choice of ν (see Mawhin [8]):

$$p + m^2 < 2\nu < q + (m + 1)^2.$$

Now Banach's fixed point theorem is applied; replacing the arbitrary $y \in H$ by a (2π)-periodic function $e(t) \in C^0$ the uniquely determined fixed point of (10) becomes a periodic solution in the classical sense.

Moreover, consider the vector differential equation

$$x'' + Cx' + \operatorname{grad} G(x) = e(t) \equiv e(t + 2\pi), \; x \in \mathbf{R}^n \tag{12}$$

where $G(x) \in C^2(\mathbf{R}^n)$,

$H(x) = (G_{x_i x_k}) = H^*(x) \in C^0$,

$e(t) \in C^0(\mathbf{R})$,

$C = \operatorname{diag}(c_1, \cdots, c_n)$, c_i real constants.

The oscillation problem can be solved in the same way as in case $n = 1$:

In the Hilbert space $(H)^n$ with the inner product $(x_1, y_1) + \cdots + (x_n, y_n)$ the differential operator and its inverse are determined by $\operatorname{col}(A_1 x_1, \cdots, A_n x_n)$ and $\operatorname{col}(A_1^{-1} y_1, \cdots, A_n^{-1} y_n)$, respectively. Here A_i denotes the operator A of Theorem 1 in case $c = c_i$.

The inverse operator has the properties discussed above; evidently, its norm is equal to $\max(\| A_1^{-1} \|, \cdots, \| A_n^{-1} \|)$. The operator originating in the nonlinear term of the differential equation can be shown to possess a Gâteaux derivative; an estimate analogous to (9), with the same value γ, can be derived via Lagrange's formula (see Vainberg [9], and compare Mawhin [8]) if we propose that $qI \leqq H(x) \leqq pI \; \forall x \in \mathbf{R}^n$ ($I : n \times n$ unit matrix).

Therefore Mawhin's theorem can be generalized as follows:

THEOREM 2. *Equation* (12) *with an arbitrary real diagonal matrix* C *has one and only one* (2π)*-periodic solution if*

$$m^2 I < qI \leqq H(x) \leqq pI < (m+1)^2 I \tag{13}$$

(m *a nonnegative integer*).

COROLLARY. *The assertion of the theorem holds, too, in the more general case when*

$$C = C^* \quad \text{an arbitrary real } n \times n \text{ matrix.}$$

Determine an orthogonal matrix Q $(Q^* = Q^{-1})$ such that

$$QCQ^* = D \quad \text{(diagonal matrix)}$$

and transform

$$y = Qx.$$

Then equation (12) is transformed into

$$y'' + Dy' + \operatorname{grad}_y G(Q^* y) = Q\, e(t)\,; \tag{14}$$

note that

$$\operatorname{grad}_y G = Q \operatorname{grad}_x G\,,$$

$$(G_{y_i y_k}) = Q\,(G_{x_i x_k})\, Q^*.$$

An immediate consequence of the last formula is that the Hessian matrix of $G(Q^* y)$ satisfies condition (13). Thus, Theorem 2. is valid in case of equation (14).

In case $c = 0$ of equation (1) Mawhin has indicated an existence theorem (to be proved via Schauder's fixed point theorem) when a weaker version of condition (2) is supposed.

THEOREM 3. *Equation* (1) *admits at least one* (2π)*-periodic solution if*

$$m^2 < q \leqq g(x)/x \leqq p < (m+1)^2 \qquad \forall x : |x| \geqq h\,. \tag{15}$$

We give an outline of the proof starting from the operator equation (8) where $y \in \mathrm{H}$ is arbitrarily chosen. The equation being solved it is clear how to proceed in order to establish the assertion of Theorem 3.

To begin with define

$$g^*(x) = \begin{cases} g(x), & |x| \geqq h \\ (x/h)\, g(h) \operatorname{sgn} x, & |x| \leqq h \end{cases}$$

and

$$\sigma(x) = g(x) - g^*(x) \quad [|\sigma(x)| \leqq \delta].$$

Hence

$$|\nu x - g(x)| \leqq \gamma |x| + \delta \qquad \text{(linear boundedness)},$$

and $x(t) \mapsto (\mathrm{B}x)(t) \equiv \nu x(t) - g(x(t))$ is a mapping of the Hilbert space H into itself for which

$$\|\mathrm{B}x\| \leqq \gamma \|x\| + \delta. \tag{16}$$

This mapping is continuous. Assume that $\mathrm{B}x$ is not continuous at $x_0 \in \mathrm{H}$. Then there is a positive number ε_0 and a sequence $\{x_k\} \subset \mathrm{H}$ such that

$$\|x_k - x_0\| \to 0 \ (k \to \infty), \ \|g_k - g_0\| \geqq \varepsilon_0 \ \forall k \in \mathbf{N}$$

where the abbreviation $g(t) = -(\mathrm{N}x)(t) \equiv g(x(t))$ is used. Without loss of generality we can assume that

$$x_k(t) \to x_0(t) \quad (k \to \infty) \qquad \text{almost everywhere on } [0, 2\pi].$$

The same is true with the sequence $\{g_k(t)\}$:

$$g_k(t) \to g_0(t) \quad (k \to \infty) \quad \text{a.e.}$$

Taking into account that the Lebesgue integral is absolutely continuous, and applying Egorov's theorem (see Hewitt–Stromberg [2]) as well as Lebesgue's dominated convergence theorem we can show that $\|g_k - g_0\| \to 0 \ (k \to \infty)$ in contrast with the assumption.

Since A^{-1} is a compact operator the mapping (10) is completely continuous on each bounded subset of H. A closed ball $\overline{\mathrm{S}}_r$ is mapped into itself when r sufficiently large:

$$r \geqq \gamma \|\mathrm{A}^{-1}\| r + \|\mathrm{A}^{-1}\| (\|y\| + \delta), \ \gamma \|\mathrm{A}^{-1}\| < 1.$$

According to Schauder's theorem it contains at least one fixed point which is a solution of (8).

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