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Upper Bounds for Conjugate Points of Nonselfadjoint Fourth Order Differential Equations

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Equazioni differenziali ordinarie. — *Upper Bounds for Conjugate Points of Nonselfadjoint Fourth Order Differential Equations.* Nota di

ALLAN EDELSON e KURT KREITH, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Nella Nota presente sono dati limiti superiori per la mutua distanza di due punti coniugati consecutivi competenti ad un'equazione differenziale ordinaria del quarto ordine non autoaggiunta.

In [1] criteria are established which assure the existence of conjugate points of the real nonselfadjoint fourth order equation

$$lu \equiv (p_2(t)\,u'' - q_2(t)\,u')'' - (p_1(t)\,u' - q_1(t)\,u)' + p_0(t)\,u = 0 \qquad (p_2(t) > 0)\,, \tag{1}$$

where by the conjugate point $\eta_1(\alpha)$ we mean the smallest $\beta > \alpha$ such that the conditions

$$u(\alpha) = u'(\alpha) = 0 = u(\beta) = u'(\beta) \tag{2}$$

are realized by some nontrivial solution $u(t)$ of (1). These criteria, however, only assure the existence of $\eta_1(\alpha)$ without providing specific upper bounds. It is the determination of upper bounds for $\eta_1(\alpha)$ which concerns us below.

The coefficients $p_k(x)$ and $q_k(x)$ are assumed real and of class C^k in an interval $[\alpha, \infty)$ with $p_2(x) > 0$. Then, as shown in [2], it is possible to represent (1) in the form

$$\begin{aligned} y'' &= a(t)\,y + b(t)\,x \\ x'' &= c(t)\,y + d(t)\,x \end{aligned} \tag{3}$$

where the $b(t) > 0$ and the coefficients of (3) are continuous in $[\alpha, \infty)$. The criteria of [1] for the existence of $\eta_1(\alpha)$ are as follows:

(i) $c(t) \geq a(t) > 0$;

(ii) $b(t) \geq d(t) > 0$;

(iii) $v'' + [\min\{b(t) - d(t), c(t) - a(t)\}]\,v = 0$ is oscillatory at $t = \infty$;

(iv) $\int^{\infty} tb(t)\,dt = \infty$:

(v) $\int^{\infty} tc(t)\,dt = \infty$.

(*) Nella seduta 10 maggio 1975.

Specifically, if (i)-(v) are satisfied, then there is a solution $(x(t), y(t))$ of (1) which satisfies

$$y(\alpha) = y'(\alpha) = 0 \quad ; \quad x(\alpha) = 1 \quad ; \quad x'(\alpha) = -v_0 < 0 ;$$
$$y(\beta) = y'(\beta) = 0 \quad ; \quad x(\beta) > 0 ;$$
$$y(t) > 0 \qquad \text{for} \quad \alpha < t < \beta .$$

Such a trajectory can be decomposed into three components:

I) A path in the first quadrant which satisfies $y(t) \geq 0$, $y'(t) \geq 0$, $0 \leq x(t) \leq 1$ and $-v_0 \leq x'(t) \leq 0$ for $\alpha \leq t \leq \beta$ and $y(\alpha) = y'(\alpha) = 0$.

II) A path in the second quadrant which enters and leaves through the positive y-axis at $t = \beta$ and $t = \gamma$, respectively.

III) A path in the first quadrant which satisfies $y(t) \geq 0$, $y'(t) \leq 0$, $x(t) \geq 0$, $x'(t) \geq 0$ for $\gamma \leq t \leq \eta_1(\alpha)$ and $y(\eta_1(\alpha)) = y'(\eta_1(\alpha)) = 0$.

Our upper bounds for $\eta_1(\alpha)$ will depend on establishing bounds for $\beta - \alpha$, $\gamma - \beta$, and $\eta_1(\alpha) - \gamma$ in terms of the coefficients of (3). It is assumed that the conditions (i)-(v) assuring the existence of $\eta_1(\alpha)$ are satisfied throughout.

Path I. Upper bounds for $\beta - \alpha$ will be established in two steps. We consider first the case where v_0 is large and seek a pair (v_0, β_1) such that

(i) $0 < x(t) \leq 1$ for $\alpha \leq t \leq \beta_1$ leads to a contradiction;

and

(ii) for every δ satisfying $\alpha \leq \delta \leq \beta_1$, $0 \leq x(t) \leq 1$ for $\alpha \leq t \leq \delta$ implies $x'(t) \leq 0$ for $\alpha \leq t \leq \delta$.

The first condition assures that the trajectory cannot remain in the strip $0 \leq x \leq 1$ beyond $t = \beta_1$ while the second assures that it cannot leave the strip across the lin $x = 1$. Thus $\beta_1 - \alpha$ will be an upper bound for Path I when v_0 is sufficiently large.

To establish (i) we integrate the first equation of (3) to note that if $x(t) \leq 1$, then

$$y'(t) = \int_\alpha^t (ay + bx)\, ds \leq \int_\alpha^t (ay + b)\, ds .$$

Defining

$$A(t) = \int_\alpha^t a(s)\, ds \quad , \quad B(t) = \int_\alpha^t b(s)\, ds , \qquad \text{etc.,}$$

we integrate by parts to get

$$0 \leq y'(t) \leq B(t) + A(t)\, y(t) - \int_\alpha^t A y'\, ds$$

along I, so that

$$y' \leq B + Ay .$$

In order to get and upper bound for $y(t)$ we let $Y(t)$ denote the solution of

$$Y' = B + AY$$
$$Y(\alpha) = 0$$

and use standard comparison theorems to conclude that $y(t) \leq Y(t)$ for $\alpha \leq t \leq \beta_1$. Turning now to the second equation in (3) we write

$$x'(t) = -v_0 + \int_\alpha^t (cy + dx)\, ds$$

$$\leq -v_0 + \int_\alpha^t cY\, ds + D(t)$$

so that $x'(t) \leq 0$ for $\alpha \leq t \leq \beta_1$ whenever

$$v_0 \geq \int_\alpha^{\beta_1} c(t)\, Y(t)\, dt + D(\beta_1) . \tag{4 a}$$

We also have

$$x(t) \leq 1 - v_0(t - \alpha) + \int_\alpha^t \int_\alpha^s cY\, dr\, ds + \int_\alpha^t D\, ds .$$

This last inequality contradictis the assumption $x(t) > 0$ for $\alpha \leq t \leq \beta_1$ in case

$$v_0 \geq \frac{1 + \int_\alpha^{\beta_1} \int_\alpha^t c(s)\, Y(s)\, ds\, dt + \int_\alpha^{\beta_1} D(t)\, dt}{\beta_1 - \alpha} . \tag{4 b}$$

We therefore fix a pair $(\tilde{v}_0, \beta_1)$ satisfying (4 a) and (4 b) above, accept β_1 as a possible upper bound for β in case $v_0 \geq \tilde{v}_0$, and go on to consider the case $v_0 < \tilde{v}_0$. Trajectories satisfying $v_0 < \tilde{v}_0$ will remain in the first quadrant for $\alpha \leq t \leq \alpha + \frac{1}{\tilde{v}_0}$. Fixing $\tilde{\alpha}$ satisfying $\alpha < \tilde{\alpha} \leq \alpha + \frac{1}{\tilde{v}_0}$, we clearly have $x(\tilde{\alpha}) > 0$, $y(\tilde{\alpha}) > 0$, $x'(\tilde{\alpha}) > -v_0$ and

$$x(t) \geq 1 - \tilde{v}_0(t - \alpha) > 0$$

for $\alpha \leq t \leq \tilde{\alpha}$ so that

$$y'(\tilde{\alpha}) \geq \int_\alpha^{\tilde{\alpha}} bx\, dt \geq \int_\alpha^{\tilde{\alpha}} b[1 - \tilde{v}_0(t - \alpha)]\, dt > 0 .$$

Denoting the last integral above by k, we have $y(t) \geq k(t - \tilde{\alpha})$ for $\tilde{\alpha} \leq t \leq \beta$. Going back to (3) again we have for $t \geq \tilde{\alpha}$

$$x'(t) \geq x'(\tilde{\alpha}) + \int_{\tilde{\alpha}}^{t} cy \, ds$$

$$\geq - \tilde{v}_0 + k \int_{\tilde{\alpha}}^{t} (s - \tilde{\alpha}) \, c \, ds .$$

Since a trajectory which realizes $\eta_1(\alpha)$ has a zero in $x(t)$ before a zero in $x'(t)$, the inequality

$$\tilde{v}_0 \leq k \int_{\tilde{\alpha}}^{\beta_2} (t - \alpha) \, c(t) \, dt \tag{5}$$

determines an upper bound for β in case $v_0 \leq \tilde{v}_0$. Combining the two possible cases we have max $[\beta_1, \beta_2]$ as an upper bound for β, where β_1 is determined by (4 a) and (4 b), while β_2 is determined by (5)

Path II. We now consider the problem of estimating the time, $\gamma - \beta$, that the trajectory remains in the second quadrant. Let II denote the open second quadrant, $\mathrm{II} = \{(x, y) ; x < 0, y > 0\}$, and we recall the following definitions and recults from [1]. Define

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad , \quad A(t) = \begin{pmatrix} d(t) & c(t) \\ b(t) & a(t) \end{pmatrix} \quad , \quad H = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then for $Y \in \mathrm{II}$, we have

$$\begin{aligned} &\langle H, Y \rangle = y - x > 0 \\ &- \langle H, AY \rangle = (b - d)(-x) + (c - a) y \geq m (y - x) \geq 0, \end{aligned} \tag{6}$$

where $m(t) = \min \{ b(t) - d(t) \ , \ c(t) - a(t) \}$. Clearly $- \dfrac{\langle H, AY \rangle}{\langle H, Y \rangle} \geq m$. By (iii) the equation

$$v'' + m(t) v = 0 \tag{7}$$

is oscillatory, and we let $t_1 < t_2 < \cdots$ be the zeros of a solution, where $t_i \uparrow \infty$. Let t_k be the least t_i such that $\beta \leq t_k$. A direct calculation yields

$$\frac{d}{dt} \left[uu' - u^2 \frac{\langle H, Y' \rangle}{\langle H, Y \rangle} \right] = - mu^2 - u^2 \frac{\langle H, AY \rangle}{\langle H, Y \rangle} + \left[u' - u \frac{\langle H, Y' \rangle}{\langle H, Y \rangle} \right]^2,$$

and by integration we obtain

$$0 = \left[uu' - u^2 \frac{\langle H, Y' \rangle}{\langle H, Y \rangle} \right]_{t_k}^{t_{k+1}} \geq - \int_{t_k}^{t_{k+1}} \left[m + \frac{\langle H, AY \rangle}{\langle H, Y \rangle} \right] u^2 \, dt$$

with equality if and only if $u(t) \equiv \langle H, Y(t) \rangle$ in (t_k, t_{k+1}). If $Y(t)$ remains in II for all t in (t_k, t_{k+1}), then $\langle H, Y(t) \rangle > 0$ in (t_k, t_{k+1}), and the above inequality must be strict. This contradicts (6) which means $Y(t)$ cannot remain in II in (t_k, t_{k+1}). Setting $\beta = t_k$, an upper bound for $\gamma - \beta$ is then given by

$$\gamma - \beta < t_{k+1} - t_k . \tag{8}$$

For example, if $m(t) \geq k^2 > 0$, then by the Sturm comparison theorem

$$\gamma - \beta < \frac{\pi}{k} .$$

Path III. To estabish upper bounds for $\eta_1(\alpha) - \gamma$ we begin by re-normalizing the problem with the assumption $y(\gamma) = 1$. It is clear that any conjugate point trajectory re-enters the first quadrant with initial velocity components $x'(\gamma) = v_x > 0$ and $y'(\gamma) = -v_y < 0$. Letting $\theta = \cot^{-1}(v_y/v_x)$ we consider separately the cases θ large and θ small.

The third component of a conjugate point trajectory satisfies $0 < x(t)$ and $0 \leq y(t)$ for all $t > \gamma$. Hence from the positivity of the coefficients $a(t), b(t), c(t)$, and $d(t)$ in (3) we have $0 < x''(t)$ and $0 < y''(t)$, and it follows that $v_x(t - \gamma) \leq x(t)$ for all $t \geq \gamma$. Integrating the first equation of (3) yields

$$\begin{aligned} y'(t) &= -v_y + \int_\gamma^t ay + bx \\ &\geq -v_y + \int_\gamma^t b(s)\, v_x(s - \gamma)\, ds. \end{aligned} \tag{9}$$

Along the third component of a conjugate point trajectory $y'(t) \geq 0$ implies that $y \geq \eta_1(\alpha)$. Therefore if

$$v_y \leq v_x \int_\gamma^t (s - \gamma)\, b(s)\, ds \tag{10}$$

then $t \geq \eta_1(\alpha)$. In other words the inequality

$$\cot \theta_0 \leq \int_\gamma^{\delta_1} (t - \gamma)\, b(t)\, dt \tag{11}$$

determines an upper bound δ_1 for $\eta_1(\alpha)$ in case $\theta_0 \leq \theta \leq \frac{\pi}{2}$. We note that δ_1 is independent of the initial velocity $v = \sqrt{v_x^2 + v_y^2}$ with which the trajectory re-enters the first quadrant.

If $\delta_1(\theta_0)$ denotes the upper bound for $\eta_1(\alpha)$ obtained from (11), then $\delta_1(\theta) \to \infty$ as $\theta_0 \downarrow 0$. We therefore fix an appropriate pair (θ_0, δ_1), restrict our attention to the case $\theta < \theta_0$, and consider the two case v_y large and v_y small.

In case v_y is sufficiently large, the trajectory will enter the fourth quadrant for sufficiently large values of t. To see this we use (3), the condition $0 \leq y(t) \leq 1$ and an integration by parts to write

$$x'(t) \leq v_x + \dot{C}(t) + D(t)x - \int_{\gamma}^{t} D(s)\, x'(s)\, ds$$

$$\leq y_y \tan \theta_0 + C(t) + D(t)\, x(t)$$

for $\gamma \leq t \leq \eta_1(\alpha)$. If $X(t)$ is the solution of the linear initial value problem

$$X' = v_y \tan \theta_0 + C(t) + D(t)\, X$$

$$X(\gamma) = 0$$

then $x(t) \leq X(t)$ for $\gamma \leq t \leq \eta_1(\alpha)$. Integrating (3) again yields

$$y'(t) \leq -v_y + A(t) + \int_{\gamma}^{t} b(s)\, X(s)\, ds \tag{12 a}$$

and

$$y(t) \leq 1 - v_y(t-\gamma) + \int_{\gamma}^{t} A(s)\, ds + \int_{\gamma}^{t}\int_{\gamma}^{\tau} b(s)\, X(s)\, ds\, d\tau \tag{12b}$$

for $\gamma \leq t \leq \eta_1(\alpha)$.

We now seek to rule out large values of v_y corresponding to $\theta < \theta_0$ by showing that $y(t)$ becomes negative while $y'(t)$ remains negatve, contradicting the assumption that $x(t), y(t)$ is a conjugate point trajectory. This follows from (12a) and (12b) in case

$$v_y \geq A(\delta_2) + \int_{\gamma}^{\delta_2} b(t)\, X(t)\, dt \tag{13a}$$

and

$$v_y \geq \frac{1}{\delta_2 - \gamma}\left[1 + \int_{\gamma}^{\delta_2} A(t)\, dt + \int_{\gamma}^{\delta_2}\int_{\gamma}^{t} b(s)\, X(s)\, ds\, dt\right]. \tag{13 b}$$

We therefore choose a pair $(\tilde{v}_y, \delta_2)$ satisfying (13a) and (13b) and conclude that $\eta_1(\alpha) \leq \delta_2$ whenever $v_y \geq \tilde{v}_y$. We remark that since $X(t)$ depends on θ_0, $\tilde{v}_y$ has the same dependence. We now assume that a pair $(\theta_0, \tilde{v}_y(\theta_0))$ have been fixed as above and seek to find an upper bound for $\eta_1(\alpha) - \gamma$ in the case of conjugate point trajectories satisfying $v_y \leq \tilde{v}_y(\theta_0)$.

This final estimate proceeds as the estimate for Path I when v_0 is small. Specifically we choose $\tilde{\gamma}$ satisfying $\gamma < \tilde{\gamma} \leq \gamma + \frac{1}{\tilde{v}_y}$ so that $\tilde{\gamma} < \eta_1(\alpha)$. Then we clearly have

$$y(\tilde{\gamma}) > 0 \quad , \quad x(\tilde{\gamma}) > 0 \quad , \quad y'(\tilde{\gamma}) > -\tilde{v}_y$$

and

$$y(t) \geq 1 - \tilde{v}_y (t - \gamma) > 0$$

for $\gamma \leq t \leq \tilde{\gamma}$. Therefore

$$x'(\tilde{\gamma}) \geq \int_{\gamma}^{\tilde{\gamma}} cy \, dt \geq \int_{\gamma}^{\tilde{\gamma}} c[1 - \tilde{v}_y (t - \gamma)] \, dt > 0 .$$

Denoting the last integral above by K we have $x(t) \geq K(t - \tilde{\gamma})$ for $\tilde{\gamma} \leq t \leq \eta_1(\alpha)$. Since $y'(t) \geq 0$ implies that $t \geq \eta_1(\alpha)$ and (3) yields

$$y'(t) \geq y'(\tilde{\gamma}) + \int_{\tilde{\gamma}}^{t} c(s) \, x(s) \, ds$$

$$> - \tilde{v}_y + K \int_{\tilde{\gamma}}^{t} (s - \tilde{\gamma}) \, c(s) \, ds .$$

The inequality

$$\tilde{v}_y \leq K \int_{\tilde{\gamma}}^{\delta_3} (t - \tilde{\gamma}) \, c(t) \, dt$$

determines a δ_3 which is an upper bound for $\eta_1(\alpha)$ in case $v_y \leq \tilde{v}_y$. In any case $\max \{ \delta_1(\theta_0), \delta_2, \delta_3 \}$ provides an upper bound for $\eta_1(\alpha)$.

This discussion can be summarized by the following set of instructions for computing an upper bound for $\eta_1(\alpha)$:

Path I. Let $Y(t)$ denoe the solution of

$$Y' = B + AY \quad ; \quad Y(\alpha) = 0 .$$

Choose a pair $(\tilde{v}_0, \beta_1)$ satisfying

(4 a)
$$v_0 \geq \int_{\alpha}^{\beta_1} cY \, dt + D(\beta_1)$$

(4 b)
$$v_0 \geq \frac{1 + \int_{\alpha}^{\beta_1} \int_{\alpha}^{t} cY \, ds \, dt + \int_{\alpha}^{\beta_1} D \, dt}{\beta_1 - \alpha} .$$

Choose $\tilde{\alpha}$ satisfying $\alpha < \tilde{\alpha} \leq \alpha + \frac{1}{\tilde{v}_0}$, define

$$k = \int_{\alpha}^{\tilde{\alpha}} b[1 - \tilde{v}_0 (t - \alpha)] \, dt > 0$$

and find β_2 satisfying

$$\tilde{v}_0 \leq k \int_{\tilde{\alpha}}^{\beta_2} (t - \alpha)\, c(t)\, dt.$$

Choose $\beta = \max\ [\beta_1, \beta_2]$.

Path II. Define $m(t) = \min\{b(t) - d(t), c(t) - a(t)\}$. Let $u(x)$ be a nontrivial solution of

$$u'' + m(t)\, u = 0$$

$$u(\beta) = 0 .$$

Choose γ to be any upper bound for the first zero of $u(x)$ to the right of $x = \beta$.

Path III. Choose a pair (θ_0, δ_1) satisfying

$$\cot \theta \leq \int_{\gamma}^{\delta_1} (t - \gamma)\, b(t)\, dt. \tag{11}$$

Let $X(t)$ denote the solution of

$$X' = v_y \tan \theta_0 + C(t) + D(t)\, X \quad ; \quad X(\gamma) = 0$$

and choose a pair $(\tilde{v}_y, \delta_2)$ satisfying

$$v_y \geq A(\delta_2) + \int_{\gamma}^{\delta_2} bX\, ds \tag{13 a}$$

$$v_y \geq \frac{1}{\delta_2 - \gamma}\left[1 + \int_{\gamma}^{\delta_2} A\, dt + \int_{\gamma}^{\delta_2}\int_{\gamma}^{t} bX\, ds\, dt\right]. \tag{13 b}$$

Then choose $\tilde{\gamma}$ satisfying $\gamma < \tilde{\gamma} < \gamma + \frac{1}{\tilde{v}_y}$, define

$$K = \int_{\gamma}^{\tilde{v}} c\,[1 - \tilde{v}_y (t - \gamma)]\, dt > 0$$

and determine δ_3 satisfying

$$\tilde{v}_y \leq K \int_{\tilde{\gamma}}^{\delta_3} t c(t)\, dt.$$

Then $\eta_1(\alpha) - \gamma \leq \max\ [\delta_1, \delta_2, \delta_3]$

The above discussion does not determine the size of the error of a particular estimate for $\eta_1(\alpha)$ nor whether there is an optimal way in which to apply this procedure. We can however consider an example of an equation for which $\eta_1(\alpha)$ is known and thereby obtain some feeling for these questions.

For the equation $y^{iv} = y$ one can readily calculate that $\eta_1(\alpha) - \alpha \approx \approx 4.72$ for all α. Choosing $\alpha = -2.36$ we have $\eta_1(\alpha) \approx 2.36$ attained by $y \approx -7.5 \cos 2.36\, x + \cosh 2.36\, x$. By calculating y'' we see that for this equation, Path I and Path III are of approximate duration 1.1 while Path II has duration of about 2.6, Our procedure applied to this equation is as follows:

Path I. Choosing $\alpha = 0$ we get $Y(t) = \frac{1}{2} t^2$. Choose $\beta_1 = 1.65$ and determine $\tilde{v}_0$ by

$$v_0 \geq \int_0^{1.65} \tfrac{1}{2} t^2 \, dt \approx .68$$

$$v_0 \geq \frac{1}{1.65} \left(1 + \int_0^{1.65} \frac{t^3}{6} \, dt \right) \approx .79$$

so that we way choose $\tilde{v}_0 = .79$. We next choose $\tilde{\alpha} = 1.20$ satisfying $0 < \tilde{\alpha} \leq \frac{1}{.79}$ and compute

$$k = \int_0^{1.20} (1 - .79t) \, dt = .63 .$$

Then β_2 is determined by $.79 \leq .63 \int_{1.20}^{\beta_2} t \, dt$ or $\beta_2 \geq 1.99$. Thus $\beta \leq 1.99$.

Path II. $m(t) \equiv 1$ so that an upper bound for Path II is π. Therefore $\gamma \leq 1.99 + 3.14 = 5.13$.

Path III. Since we have constant coefficients we can translate the problem to $\gamma = 0$. We determine δ_1 and θ_0 by setting $\cot \theta = 2$ so that by (11)

$$2 \leq \int_0^{\delta_1} t \, dt$$

which yields $\delta_1 = 2$. Then we get

$$X(t) = \tfrac{1}{2} v_y t + \tfrac{1}{2} t^2$$

and choose $(\tilde{v}_y, \delta_2)$ so that

$$v_y \geq \int_0^{\delta_2} \left(\frac{1}{2} v_y t + \frac{1}{2} t^2 \right) dt \geq \frac{1}{4} v_y \delta_2^2 + \frac{1}{6} \delta_2^3$$

and

$$v_y \geq \frac{1}{\delta_2}\left[1 + \int_0^{\delta_2}\int_0^t \left(\frac{1}{2} v_y s + \frac{1}{2} s^2\right) ds\right]$$

$$\geq \frac{1}{\delta_2}\left[1 + \frac{\delta_2^3}{12} v_y + \frac{\delta_2^4}{24}\right].$$

Choosing $\delta_2 = 1$ implies that we may use $\tilde{v}_y = 1.14$.
Choosing $\tilde{\gamma} = .75$ satisfying $\theta \leq \tilde{\gamma} \leq \frac{1}{\tilde{v}_y}$, we define

$$K = \int_0^{.75} (1 - 1.14\, t)\, dt \approx .43$$

and determine δ_3 to satisfy

$$1.14 \leq .43 \int_{.75}^{\delta_3} t\, dt$$

by choosing $\delta_3 = 2.42$. This yields $\eta_1(\alpha) - \gamma \approx 2.4$ and $\eta_1(\alpha) - \alpha \approx 7.55$, compared with the values 4.72 obtained from precise knowledge of the solution which attains the conjugate point. As is to be expected, the worst error occurs in Paths I and III.

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