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Constrained Kullback-Leibler Estimation; Generalized Cobb-Douglas Balance, and Unconstrained Convex Programming

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Ricerca operativa. — *Constrained Kullback-Leibler Estimation; Generalized Cobb–Douglas Balance, and Unconstrained Convex Programming.* Nota di ABRAHAM CHARNES e WILLIAM W. COOPER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si dà una caratterizzazione completa delle relazioni tra: (1) un caso più generale della stima Kullback–Leibler con una distribuzione discreta e finita a vincoli lineari di disuguaglianza; (2) una minimizzazione non vincolata di un potenziale convesso, oppure la negativa della funzione utilità; (3) le equazioni generalizzate Cobb–Douglas di «equilibrio» o di «bilancia contabile». Inoltre, si ottiene una caratterizzazione in termini di una coppia esattamente duale per una classe di problemi di programmazione geometrica estesa, in luogo delle più deboli condizioni necessarie o sufficienti di Duffin, Peterson e Zener. Si presenta infine una nuova classe di soluzioni «entropiche» per funzioni caratteristiche di giuochi con n–persone, che ammette una caratterizzazione equivalente al duale di una programmazione convessa non vincolata.

0. INTRODUCTION

In [1] it was shown that a nonlinear system of equations used for estimation of interzonal transfers in traffic engineering and marketing were in fact derivable from an extremal principle of the Kullback-Leibler type [2] of information-theoretic, or entropic, statistical estimation. In [3] it was shown that the accounting balance equations for a cartel or "resource-value transfer" economy could be derived on the one hand from unconstrained minimization of an "economic" potential function, or dually, from Kullback-Leibler statistical estimates constrained by a linear inequality system of pure "network" or "distribution" type.

The work of Akaike [4] and others [5] has shown that both the principle of maximum likelihood and the Fisher-information approach are asymptotically equivalent to Kullback-Leibler estimation, thus yielding on the one hand a statistical decision theoretic interpretation of maximum likelihood and on the other hand a single rational decision theoretic method for statistical estimation, statistical hypothesis testing, and spectral analysis of time series with an objective designation of the number of terms to be carried [4].

In this paper we establish the new mathematical "troika" or "triality" (rather than "duality") now apparent. Thus we characterize completely the relationships of (1) a more general case tan Kullback-Leibler estimation with finite discrete distributions and linear inequality constraints, (2) unconstrained minimization of a convex potential, or neg-utility function and (3) generalized Cobb-Douglas "equilibrium" or "accounting balance" equations. In so doing, we present an exact and sharp "MECE" (mutually exclusive and collectively exhaustive) characterization of duality for the relevant class of ***extended*** geometric programming problems [5] similar to the Charnes-Cooper linear programming duality characterization [6] of four MECE dual pairs from nine possibilities, and the Ben-Israel, Charnes,

(*) Nella seduta del 12 aprile 1975.

Kortanek [7] general convex programming characterization of eleven MECE pairs from a possible 49. Here there are precisely three MECE pairs.

As is developed elsewhere by Charnes, Haynes, Phillips, the extended geometric programming formulation yields an unconstrained utility theory of interzonal transfers in transportation and the classic traffic engineers' estimate. This approach differs from and is more general than the ordinary geometric programming approach of utility theorists Beckmann and Golob.

We also present a new solution concept for n-person cooperative games which relates the Charnes-Kortanek convex nucleus notion [8] to one of Kullback-Leibler estimation type, and dually to an unconstrained convex extremal principle. Further developments are being pursued elsewhere.

Statistical interpretation of the new unconstrained dual problem to Kullback-Leibler estimation herein presented and its extension to general distributions (and continuous programming problems) is under way by Charnes, Cooper and Ben-Tal.

1. EXISTENCE OF INFIMA AND MINIMA

Consider the unconstrained convex programming problem

$$\text{(1.1)} \qquad \min \mathscr{C}(z) \equiv c^{\mathrm{T}} e^{\mathrm{A}z} - b^{\mathrm{T}} z\,, \qquad \text{where } c^{\mathrm{T}} > 0\,,$$

$$\text{and} \quad e^{\mathrm{A}z} \equiv (\cdots, e^{({}_i\mathrm{A}z)}, \cdots)^{\mathrm{T}} \quad , \qquad \text{where } {}_i\mathrm{A} \equiv ith \text{ row of A}.$$

This substantially generalizes the problem treated by Charnes and Cooper in [3].

THEOREM 1. *$\mathscr{C}(z)$ is bounded below iff there exists $\delta^{\mathrm{T}} \geq 0$ such that $\delta^{\mathrm{T}}\mathrm{A} = b^{\mathrm{T}}$.*

Proof. $\mathscr{C}(z)$ is *not* bounded below iff there exists a sequence $\{z^n\}$ such that $\mathscr{C}(z^n) \to -\infty$. But $\mathscr{C}(z^n) \to -\infty$ iff (*a*) $\mathrm{A}z^n$ is bounded above while (*b*) $b^{\mathrm{T}} z^n \to \infty$.

Thus, considering the dual linear programming problems:

$$\text{(1.2)} \qquad \begin{array}{ll} \text{I} & \text{II} \\ \max b^{\mathrm{T}} z & \min \delta^{\mathrm{T}} a \\ \mathrm{A}z \leq a & \delta^{\mathrm{T}}\mathrm{A} = b^{\mathrm{T}} \quad , \quad \delta^{\mathrm{T}} \geq 0 \end{array}$$

where the vector "a" represents an upper bounding vector, we are in the situation in I that $\max b^{\mathrm{T}} z = \infty$. By the extended dual theorem of linear programming [9] this occurs iff II is inconsistent, i.e. there is *no* solution to $\delta^{\mathrm{T}} \geq 0$, $\delta^{\mathrm{T}}\mathrm{A} = b^{\mathrm{T}}$. Hence $\mathscr{C}(z)$ is bounded below iff there exists $\delta^{\mathrm{T}} \geq 0$ such that $\delta^{\mathrm{T}}\mathrm{A} = b^{\mathrm{T}}$.

Turning now to minimum infimum differentiation, since the solutions δ^{T} of $\delta^{\mathrm{T}} \geq 0$, $\delta^{\mathrm{T}}\mathrm{A} = b^{\mathrm{T}}$ form a convex set, there is a *unique* maximal set of rows of A, say $\mathrm{A_D}$, and some $\bar{\delta}^{\mathrm{T}} = (\bar{\delta}_{\mathrm{D}}^{\mathrm{T}}, 0)$ so that $\bar{\delta}_{\mathrm{D}}^{\mathrm{T}} > 0$ and $\bar{\delta}^{\mathrm{T}}\mathrm{A} \equiv \bar{\delta}_{\mathrm{D}}^{\mathrm{T}}\mathrm{A_D} + 0 \cdot \mathrm{A_R} = b^{\mathrm{T}}$, where $\mathrm{A_R}$ designates the remaining rows of A.

Then

$$\mathscr{C}(z) = c_D^T e^{A_D z} - \bar{\delta}_D^T A_D z + c_R^T e^{A_R z}, \qquad \text{where} \quad c^T = (c_D^T, c_R^T). \tag{1.3}$$

We note that the rows of A_R are linearly independent of the rows of A_D. For if some rows $A_L = LA_D$, for some matrix L, then we can write

$$b^T = \bar{\delta}_D^T A_D + \delta_L^T (A_L - LA_D) = (\bar{\delta}_D^T - \delta_L^T L) A_D + \bar{\delta}_L^T A_L . \tag{1.4}$$

We can pick $\delta_L^T > 0$ sufficiently small that $\bar{\delta}_D^T - \delta_L^T L > 0$. Thus A_L is part of A_D.

Suppose $\mathscr{C}(z^n) \to \inf\limits_z \mathscr{C}(z)$. If ${}_i A_D z^n$ is unbounded for any row i of A_D, we can pick a subsequence, denoted z^n b.a.o.n. (by abuse of notation), so that ${}_i A_D z^n \to$ one of $+\infty$, $-\infty$.

But then

$$\mathscr{C}_i(z^n) = c_{D_i} e^{({}_i A_D z^n)} - \bar{\delta}_{D_i}({}_i A_D z^n) \to +\infty , \tag{1.5}$$

in either case, and since $\mathscr{C}_i(z^n) \leq \mathscr{C}(z^n) + \text{const.}$, $\mathscr{C}(z^n) \to +\infty$, a contradiction. Thus for some a_D,

$$| A_D z^n | \leq a_D , \qquad \forall n . \tag{1.6}$$

We can pick a subsequence z^n (b.a.o.n.) so that

$$A_D z^n \to \bar{w}_D , \qquad \text{for some} \quad \bar{w}_D . \tag{1.61}$$

Clearly ${}_i A_R z^n$ is bounded above for all rows i of A_R, i.e. for some a_R, $A_R z^n \leq a_R$. If also ${}_i A_R z^n$ is bounded below for all rows i of A_R, then we can pick a subsequence z^n (b.a.o.n.) so that both

$$A_D z^n \to \bar{w}_D , \qquad \text{and} \quad A_R z^n \to \bar{w}_R \qquad \text{for some} \quad \bar{w}_R . \tag{1.62}$$

By the Farkas-Minkowski closure corollary [10], there exists $\bar{z}$ such that

$$A_D \bar{z} = \bar{w}_D , \; A_R \bar{z} = \bar{w}_R . \qquad \text{But} \quad \mathscr{C}(\bar{z}) = \underset{n\to\infty}{\mathrm{L}} \, \mathscr{C}(z^n) = \inf_z \mathscr{C}(z) .$$

Hence $\mathscr{C}(z)$ has a minimum at $z = \bar{z}$ if $A_R z^n$ is bounded below.

If $\mathscr{C}(z)$ has only an infimum, it is thus necessary that $A_R z^n$ be unbounded below, i.e., for *all* $p^T > 0$, $p^T A_R z^n \to -\infty$ as $n \to \infty$. Consider therefore the dual l.p. problems:

$$\begin{array}{lll} & \text{I}' & \text{II}'' \\ & \max -p^T A_R z & \min \delta_D^T a_D + \delta_R^T a_R \\ (1.7) & A_D z \leq a_D & \delta_D^T A_D + \delta_R^T A_R = -p^T A_R \;, \quad \delta_D, \delta_R \geq 0 . \\ & A_R z \leq a_R . & \end{array}$$

The primal problem I′ has $\max -p^T A_R z = \infty$ for every $p^T > 0$. This, by the extended dual theorem [9], holds iff the system

$$\delta_D^T A_D + \delta_R^T A_R = -p^T A_R \quad , \quad \delta_D , \delta_R \geq 0 \tag{1.71}$$

is inconsistent for every $p^T > 0$. The latter condition may be rendered more simply by noting (i) that $q^T = \delta_R^T + p^T$, some $\delta_R^T \geq 0$ and all $p^T > 0$ represents all $q^T > 0$ and (ii) that A_D is the *unique* maximal set of rows of A for which there exists $\bar{\delta}_D^T > 0$ such that $\bar{\delta}_D^T A_D = b^T$.

THEOREM 2. *$\mathscr{C}(z)$ has an infimum and no minimum iff*

$$(1.72)\qquad \begin{array}{ll} \text{(i)} & \delta^T A = b^T,\ \delta \geq 0 \qquad \textit{has a solution, while} \\ \text{(ii)} & \delta_D^T A_D + \delta_R^T A_R = 0 \ ,\quad \delta_D^T \geq 0 \ ,\quad \delta_R^T > 0 \qquad \textit{has no solution.} \end{array}$$

Therefore,

THEOREM 3. *$\mathscr{C}(z)$ has a minimum iff there exists a solution to*

$$(1.8)\qquad \begin{array}{ll} \text{(a)} & \delta^T A = b^T,\ \delta \geq 0 \qquad \textit{and a solution to} \\ \text{(b)} & \delta_D^T A_D + \delta_R^T A_R = 0 \quad , \quad \delta_D^T \geq 0 \ ,\quad \delta_R^T > 0 \ , \end{array}$$

where A_D is the maximal set of rows of A for which a positive solution to (a) exists.

From Theorem 3, we also have an immediate corollary which encompasses the Charnes-Cooper characterization [3] of a minimum of $\mathscr{C}(z)$ for a special class of matrices A:

COROLLARY 3. *If the entries of A are unisignant (non-negative or non-positive), then $\mathscr{C}(z)$ has a minimum iff for some submatrix A_D of rows of A of equal rank to A there exists $\delta_D^T > 0$ such that $\delta_D^T A_D = b^T$.*

Proof. Unisignance implies no solution to (b) unless $A_R = 0$.

THEOREM 3.1. *$\mathscr{C}(z)$ has a minimum iff $A_D = A$, i.e., there exists $\delta > 0$ such that $\delta^T A = b^T$.*

Proof. Suppose $\delta_D^T A_D = b^T$, $\delta_D > 0$ and $\bar{\delta}_D^T A_D + \bar{\delta}_R^T A_R = 0$, $\bar{\delta}_D \geq 0$, $\bar{\delta}_R > 0$. Then $\bar{\bar{\delta}}^T = 1/2\,[(\delta_D^T, 0) + (\bar{\delta}_D^T, \bar{\delta}_R^T)] > 0$ and $\bar{\bar{\delta}}^T A = b^T$. Contrariwise, if there exists $\delta > 0$ such that $\delta^T A = b^T$, then (a) and (b) are satisfied with $A_R = 0$.

There follows as an immediate corollary:

COROLLARY 3.1. *$\mathscr{C}(z)$ has only an infimum iff*

(a) $\delta^T A = b^T$, $\delta \geq 0$ *has a solution,*

(b) $\prod_i \delta_i = 0$ *in every solution.*

It is to be noted that in general the rows of A_D are *not* linearly independent. In fact, in the interesting case of exponential sums, i.e. $b^T = 0$, they are always linearly dependent when a minimum exists. For exponential sums our theorem becomes:

THEOREM 3.2. *$\mathscr{C}(z) = c^T e^{Az}$, $c^T > 0$, has a minimum iff for some submatrix A_D of rows of A of equal rank to A, there exists $\delta_D^T > 0$ such that $\delta_D^T A_D = 0$.*

Proof. Condition (a) of Theorem 3 is automatically satisfied. Condition (b) is now a further statement about solutions to $\delta^T A = b^T$, $\delta^T \geq 0$, since $b^T = 0$. By definition of A_D, A_R would have to be part of A_D if $A_R \neq 0$. Thus $A_R = 0$.

Geometrically, the theorem can be interpreted as follows: A minimum exists iff the origin is a *proper* interior point of the convex hull of a subset of rows of A which span the row-space of A.

2. "MECE" DUALITY

THEOREM 4. *There are three mutually exclusive and collectively exhaustive duality states of nine a priori possibilities*:

1. $\mathscr{C}(z)$ *has no lower bound and* $\delta \geq 0$, $\delta^T A = b^T$ *has no solution*;

2. $\mathscr{C}(z)$ *has only an infimum and* $\delta \geq 0$, $\delta^T A = b$ *is consistent such that* $\prod_i \delta_i = 0$ *for every solution* (*e.g.* $\delta_i = 0$, $i \in R \neq \varnothing$ *in every solution of* $\delta \geq 0$, $\delta^T A = b^T$). *Further,* $\inf \mathscr{C}(z) = \max v(\delta)$ *with* $\delta \geq 0$, $\delta^T A = b^T$ *and* $\inf \mathscr{C}(z) = \min \mathscr{C}_D(z)$, *where* $\mathscr{C}_D(z) = c_D^T e^{A_D z} - b^T z$;

3. $\mathscr{C}(z)$ *has a minimum and there exists* $\delta > 0$, $\delta^T A = b^T$. *Further*

(i) $\min \mathscr{C}(z) = \max v(\delta)$, $\delta \geq 0$, $\delta^T A = b^T$;

(ii) *if* $\mathscr{C}(z^*) = \min \mathscr{C}(z)$,

then $v(\delta^*) = \max v(\delta)$, $\delta \geq 0$, $\delta^T A = b^T$, *where* $\delta_i^* = c_i e^{({}_iAz^*)}$ *for all* i;

(iii) *if* $v(\delta^*) = \max v(\delta)$, $\delta \geq 0$, $\delta^T A = b^T$,

then $\mathscr{C}(z^*) = \min \mathscr{C}(z)$, *where* ${}_iAz^* = \ln(\delta_i^*/c_i)$ *for all* i.

Proof. Clearly the three conditions for $\mathscr{C}(z)$ and separately for the δ–system are "MECE". Duality states (1), (2), (3) are established respectively by Theorem 1, Theorem 3.1 and Corollary 3.1.

To establish the further properties of states (2) and (3), we note easily from [6] that

$$\mathscr{C}(z) \equiv c^T e^{Az} - b^T z \geq v(\delta) \equiv \delta^T e - \delta^T \ln\left(\frac{\delta}{c}\right) \tag{2.1}$$

for all $\delta^T \geq 0$, $\delta^T A = b^T$ and all z.

In state (3), since $\mathscr{C}(z)$ is convex and analytic in z, at a minimum z^*,

$$\frac{\partial \mathscr{C}}{\partial z_j^*} = \sum_i c_i e^{({}_iAz^*)} a_{ij} - b_j = 0, \qquad \forall_j. \tag{2.2}$$

Setting $\delta_i^* = c_i e^{({}_iAz^*)}$, $\delta^* > 0$ and $\delta^{*T} A = b^T$.
Further,

$$v(\delta^*) = c^T e^{Az^*} - \sum_i \delta_i^* \ln(e^{{}_iAz^*}) = c^T e^{Az^*} - \delta^{*T} Az^* = \mathscr{C}(z^*). \tag{2.3}$$

Hence $v(\delta^*) = \max v(\delta)$ for $\delta \geq 0$, $\delta^T A = b^T$.

Thus in state (3) $v(\delta)$ has a maximum. Further, since $v(\delta)$ is strictly concave in δ, it has a unique maximum at $\delta_i = \delta_i^* = c_i\, e^{(_i A z^*)}$ where z^* is any minimum for $\mathscr{C}(z)$. Thus, also, given the maximizing δ^*, there must exist a solution to $A\hat{z} = \ln\left(\frac{\delta^*}{e}\right)$, with $\hat{z}$ minimizing $\mathscr{C}(z)$.

In state (2), every $\delta^T = (\delta_D^T, \delta_R^T) \geq 0$ satisfying $\delta^T A = b^T$, must have $\delta_R^T = 0$. So for every solution δ,

$$v(\delta) = v_D(\delta_D) + v_R(\delta_R) = v_D(\delta_D) \tag{2.4}$$

where $v_D(\delta_D) = \delta_D^T e - \delta_D^T \ln\left(\frac{\delta_D}{c_D}\right)$, since $x \ln x = 0$ for $x = 0$, and continuity.

But now, letting $\mathscr{C}_D(z) \equiv c_D^T e^{A_D z} - b^T z$, we are in state (3) for $\mathscr{C}_D(z)$, $\delta_D^T \geq 0$ and $\delta_D^T A_D = b^T$. Thus $v(\delta)$ has a maximum in state (2), unique by strict concavity. Now

$$\mathscr{C}(z) = \mathscr{C}_D(z) + c_R^T e^{A_R z} = c_D^T e^{A_D z} - \bar{\delta}_D^T A_D z + c_R^T e^{A_R z}. \tag{2.5}$$

Let z_D^* make $\mathscr{C}_D(z)$ minimum. Consider the dual l.p. problems

$$\begin{array}{lll} & \text{I} & \text{II} \\ & \max b^T z & \min \delta_D^T A_D z_D^* + \delta_R^T A_R z_D^* \\ (2.6) & A_D z \leq A_D z_D^* & \delta_D^T A_D + \delta_R^T A_R = b^T \\ & A_R z \leq A_R z_D^* & \delta_D^T,\ \delta_R^T \geq 0 \end{array}$$

The optimum value of I is finite for otherwise $\mathscr{C}(z)$ is not bounded below.

Since $\delta_R^T = 0$ in every (hence optimal) solution of II, by the Extended Theorem of the Alternative (cf. [9], p. 441), some optimum solution of I, say $\hat{z}$, has $A_R \hat{z} < A_R z_D^*$. Further $A_D \hat{z} = A_D z_D^*$ and $b^T \hat{z} = b^T z_D^*$, otherwise $\mathscr{C}_D(z_D^*)$ is not minimum. Let $z_\rho = \rho(\hat{z} - z_D^*)$, $\rho > 0$.

Then

$$\begin{aligned} & b^T z_\rho = 0\ , \quad \forall \rho \\ (2.7) \qquad & A_D z_\rho = 0\ , \quad \forall \rho \\ & A_R z_\rho = \rho A_R(\hat{z} - z_D^*) \to -\infty \text{ (vector) as } \rho \to \infty\,. \end{aligned}$$

Now

$$\mathscr{C}(z) \geq \mathscr{C}_D(z^*) + c_R^T e^{A_R z} > \mathscr{C}_D(z^*)\,, \qquad \forall z. \tag{2.8}$$

Hence

$$\underset{\rho \to \infty}{\mathrm{L}}\ \mathscr{C}(z^* + z_\rho) = \mathscr{C}_D(z^*) = \inf \mathscr{C}(z)\,. \tag{2.9}$$

Thus in state (2), $\inf \mathscr{C}(z) = \max v(\delta)$ and $\inf \mathscr{C}(z) = \min \mathscr{C}_D(z)$.

The duality state is characterized by solution of the linear program:

$$\max \mu \ , \ \mu e^{T} - \delta^{T} \leq 0 \quad , \quad \delta^{T} A = b^{T} \quad , \quad \delta \geq 0 . \tag{2.10}$$

Infeasibility is state (1), $\mu^{*} = 0$ is state (2), $\mu^{*} > 0$ is state (3).

3. Generalized Cobb-Douglas Balance

In [3] Charnes and Cooper were able to show that the equations of accounting-balance for a resource-value transfer economy do in fact arise from (dual) extremal principles of extended geometric programming type. The equations involved, however, contain only non-negative coefficients and contain the (positive) variables only to the second degree. We observe they can be generated alternately as tight constraints in an ordinary geometric programming problem. W. Drews has posed the question of obtaining similar balances in which the coefficients are not restricted to be non-negative. Such systems do not conform to our observation re ordinary geometric programming constraints.

The preceding and the exponential mapping of the real line onto the positive numbers imply

THEOREM 5. *A necessary and sufficient condition that the generalized Cobb-Douglas equations*

$$\sum_{i} d_{ij} t_{1}^{a_{i1}} , \cdots , t_{n}^{a_{in}} = d_{j} \quad , \quad t_{j} > 0 \, , \quad \forall_{j}$$

arise from discrete Kullback-Leibler estimation with linear inequalities is that they be equivalent to a system

$$\sum_{i} c_{i} \, a_{ij} \, t_{1}^{a_{i1}} , \cdots , t_{n}^{a_{in}} = b_{j} \, t_{j} \, , \, c_{i} \quad , \quad t_{j} > 0 \, , \quad \forall i \, , j \, .$$

4. Entropic Solutions to N-person Cooperative Games

We note first that the "weighted" Kullback-Leibler estimation problem

$$\max - \sum_{i} w_{i} \, \delta_{i} \ln \left(\frac{\partial_{i}}{e c_{i}} \right) \ \text{with } \delta_{i} \geq 0, \ \delta^{T} A = b^{T}, \ \text{where } w_{i} > 0, \ \forall i, \tag{4.1}$$

may be reduced to the unweighted form by the transformation

$$\delta_{i}^{'} = w_{i} \, \delta_{i} \quad , \quad c_{i}^{'} = w_{i} \, c_{i} \quad , \quad {}_{i}A^{'} = (w_{i})^{-1} \, {}_{i}A \, . \tag{4.2}$$

Let g be the characteristic function and normalized so that $0 < g(S)$, its value for the coalition S, any subset of $N = \{1, 2, \cdots, n\}$. Let $x(S) = \sum_{j \in S} x_{j}$, for the imputation $(x_{1}, \cdots, x_{n})$. Let $x^{T} \equiv (\cdots, x(S), \cdots)$. The system defining the $x(S)$'s and the imputation may be written

$$x^{T} G = 0 \quad , \quad x(N) = g(N) \quad , \quad x^{T} \geq 0 \, . \tag{4.3}$$

Then we can define a class of weighted entropic solutions by

$$\max \sum_{S} w(S)\, x(S) \ln \left(\frac{x(S)}{eg(S)}\right) \tag{4.4}$$

subject to

$$x^T G = 0 \quad , \quad x(N) = g(N) \quad , \quad x^T \geq 0 \, .$$

Problem (4.4) has an unconstrained convex dual problem

$$\min \sum_{S} w(S)\, g(S) \exp\left[{}_S G z / w(S)\right] - g(N)\, z_N \tag{4.5}$$

This new class of solutions, which can be further modified by additional inequalities of (4.3) form (e.g. to include the core), is a variant of the Charnes-Kortanek convex nucleus class of solutions. Here the ratios $x(S)/g(S)$ are employed instead of the coalitional excesses $g(S) - x(S)$. But now the game solution is given by an unconstrained convex programming problem of simple analytic form!

5. KULLBACK-LEIBLER ESTIMATION

In Kullback's classic book *Information Theory and Statistics*, important consequences are taken from solution of the extremal system which for finite discrete distributions would be (4.1) whith only two columns for A (one asserting that $\sum_i \delta_i = 1$). Not only have we here greatly increased the type and extent of the information which can be used in the estimation (general linear inequalities), we also give a new simple unconstrained dual convex characterization of the unique optimal solution. And, recall, $\delta_i^* = c_i\, e^{i A z^*}$! These results should be particularly advantageous and important in treating estimation of the solution of (4.1). The latter forms the basis for most of the quantitative aspects of Kullback–Leibler or information-theoretic statistical theory.

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