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Ivar Massabó

A fixed-point theorem for single-valued mappings defined on a topological space

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Topologia. — *A fixed-point theorem for single-valued mappings defined on a topological space* (*). Nota di IVAR MASSABÒ, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Scopo di questa Nota è quello di mostrare come con la nozione di φ–coerenza, introdotta da Furi e Vignoli [7], si possano ottenere molti dei teoremi sull'esistenza del punto fisso.

INTRODUCTION

In the last few years many results have been obtained on the existence of "fixed-point" for single-valued mappings by generalizing classical theorems of Banach [1], Kannan [8, 9], Edelstein [6] and Kirk [10].

The main purpose of this paper is to give a test to assure the existence of fixed-points for mappings defined in topological spaces and to apply it showing how all classical fixed-point theorems can be derived.

The proper generality of our results lies in the fact that we obtain results concerning the existence of fixed-points by merely testing the "local" behaviour of the mappings instead of studying the full behaviour of them on the whole space. To do it, it is very helpful the notions of φ-coherence (see definition below) introduced by Furi and Vignoli [7]. Our goals are stated in Theorem 1.1 of § 1.

In § 2 we obtain as corollaries to this theorem results of Banach [1], Kannan [8, 9], Reich [12], Čirič [5], Sehgal [13], Edelstein [6] and Kirk [10].

1. DEFINITIONS AND NOTATIONS

Let X be a topological space and $(x_n)_{n \in \mathbf{N}}$, $\mathbf{N} = \{0, 1, 2, \cdots\}$, be a sequence of elements of X.

An element $x \in X$ is said to be a *limit point* of x_n if any neighbourhood of x contains all the x_n from a certain value of n onwards.

If x is a limit point of (x_n) we write $x_n \rightarrow x$.

An element $x \in X$ is said to be a *strong cluster point* of (x_n) is x if limit point of a subsequence (x_{n_i}) of (x_n).

It is an immediate consequence of the definitions that a limit point is a strong cluster point but in general the converse is not always true.

Let $T : X \rightarrow X$ be a mapping from a topological space X into itself and let $x \in X$.

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DEFINITION 1.1. *The set* $\mathcal{O}(x) = \{T^n(x) : T^0(x) = x \ , \ T^{n+1}(x) = T(T^n(x)) \ , \ n \in \mathbf{N}\}$ *is called the* orbit *at* x. *To any* $x \in X$ *we can associate the sequence* $(T^n(x))_{n \in \mathbf{N}}$ *of iterates of* T *at* x *which is called the* orbital sequence *at* x.

DEFINITION 1.2. *A strong cluster point* $z \in X$ *of* $(T^n(x))_{n \in \mathbf{N}}$ *is said to be " nice " if*

$$\text{“} T^{n_i}(x) \to z \text{”} \Rightarrow \text{“} T^{n_i+k}(x) \to T^k(z) \ k = 1, 2, \cdots \text{”}.$$

Remark 1.1. Čirič [4] introduced the notion of *orbital continuity* for mappings in metric spaces. A mapping T is said to be *orbitally continuous* on $\mathcal{O}(x)$ if whenever

z is limit point of $(T^{n_i}(x))_{i \in \mathbf{N}}$ then $T(z)$ is limit point of $(T^{n_i+1}(x))_{i \in \mathbf{N}}$.

Note that if the set of strong cluster points of an orbital sequence $(T^n(x))_{n \in \mathbf{N}}$ is nonempty then T is orbitally continuous on $\mathcal{O}(x)$ iff all strong cluster points of $(T^n(x))_{n \in \mathbf{N}}$ are nice.

We give an example of a mapping that is not orbitally continuous but it has a strong cluster point which is nice.

Example 1.1. Let Γ be the collection of all countable ordinals. Let X be the subset of Γ which contains the elements of Γ less or equal to $w_0 \cdot 2$ (w_0, the first transfinite ordinal) with the order topology.

Let $T : X \to X$ be defined as follows

$$T(x) = \begin{cases} x + 1 & \text{if } \ x < w_0 \cdot 2 \\ w_0 \cdot 2 & \text{if } \ x = w_0 \cdot 2 \end{cases}$$

T is not orbitally continuous on $\mathcal{O}(0)$ because w_0 is limit point of $(T^n(0))_{n \in \mathbf{N}} = (n)_{n \in \mathbf{N}}$ but $w_0 + 1$ is *not* limit point of $(T^{n+1}(0))_{n \in \mathbf{N}} = (n+1)_{n \in \mathbf{N}}$. Take $w_0 \cdot 2$, $w_0 \cdot 2$ is limit point of $(T^n(T^{w_0}(0)))_{n \in \mathbf{N}} = (w_0 + n)_{n \in \mathbf{N}}$ and $(T^{n+k}(T^{w_0}(0)))_{n \in \mathbf{N}} = (w_0 + n + k)_{n \in \mathbf{N}}$ converges to $T^k(w_0 \cdot 2) = w_0 \cdot 2$ for every $k \in \mathbf{N}$, therefore $w_0 \cdot 2$ is nice.

Finally note also that each continuous mapping is orbitally continuous on each orbit but the conserve is not always true. See the following example.

Example 1.2. Let $X = [-1, 1]$ and let $T : X \to X$ be defined as follows:

$$T(x) = \begin{cases} x/2 & \text{if } \ x > 0 \\ 1 & \text{if } \ x \leq 0 . \end{cases}$$

Evidently, T is orbitally continuous at $x = 0$ but T is not continuous at $x = 0$.

We want to introduce the notion of " orbitally φ-coherent " mapping and to prove a fixed-point theorem for this class of mappings. In a natural way, we extend the concept of φ-coherence and some of the results obtained by Furi and Vignoli [7]. In the next section, we will give applications of our main result.

Let X be a topological space and T be a mapping of X into itself. Further, let φ be a mapping from

$$\hat{\mathcal{O}}(x) = \mathcal{O}(x) \cup \{z : z \text{ is a strong cluster point of } (\mathrm{T}^n(x))_{n \in \mathbf{N}}\}$$

into a Hausdorff space Y.

DEFINITION 1.3. *The mapping* T *is said to be* orbitally φ-coherent *on* $\mathcal{O}(x)$, *if for each nice cluster point* z_0 *of the orbital sequence at* x, *we have* $z_0 = \mathrm{T}(z_0)$ *iff* $(\varphi(\mathrm{T}^n(z_0)))_{n \in \mathbf{N}}$ *is constant.*

Remark 1.2. We recall that Furi and Vignoli ([7], p. 197) said that a mapping $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ is *φ-coherent*, where $\varphi : \mathrm{X} \to \mathrm{Y}$ is continuous, if for any $x \in \mathrm{X}$, $x \neq \mathrm{T}(x)$, there exists $k \in \mathbf{N}$ such that $\varphi(x) \neq \varphi(\mathrm{T}^k(x))$ or equivalently

$$x = \mathrm{T}(x) \text{ iff } (\varphi(\mathrm{T}^n(x)))_{n \in \mathbf{N}} \text{ is constant.}$$

DEFINITION 1.4. *The mapping* φ *is said to be* orbitally continuous at a nice cluster point z *of* $(\mathrm{T}^n(x))$ *if*

$$\text{“}\mathrm{T}^{n_i}(x) \to z\text{”} \Rightarrow \text{“}\varphi(\mathrm{T}^{n_i+k}(x)) \to \varphi(\mathrm{T}^k(z))\ k = 0, 1, 2, \cdots\text{”}.$$

We want to remark that

T is orbitally continuous on $\mathcal{O}(x)$ in the sense of Čirič (Remark 1.1) if and only of any strong cluster point of $(\mathrm{T}^n(x))$ is nice. Moreover if z is a nice cluster point of $(\mathrm{T}^n(x))$ then T is orbitally continuous at z (in the sense of Definition 1.4).

We prove the following main theorem.

THEOREM 1.1. *Let* $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ *be a mapping of a topological space* X *into itself, and* φ *be a mapping of* $\hat{\mathcal{O}}(x)$ *into a Hausdorff space* Y. *If,*

(i) *T is orbitally φ-coherent on* $\mathcal{O}(x)$,

(ii) *the sequence* $(\varphi(\mathrm{T}^n(x)))_{n \in \mathbf{N}}$ *converges,*

(iii) *z is a nice cluster point of* $(\mathrm{T}^n(x))$, *and*

(iv) *φ is orbitally continuous at z*

then, $z = \mathrm{T}(z)$, *i.e.* T *has a fixed point in* X.

Proof. Because z is a nice cluster point of $(\mathrm{T}^n(x))$, z is limit point of $(\mathrm{T}^{n_i+k}(x))_{i \in \mathbf{N}}$ for each $k \in \mathbf{N}$. Since $(\varphi(\mathrm{T}^n(x)))_{n \in \mathbf{N}}$ converges to a unique point $y \in \mathrm{Y}$, by the orbital continuity of φ at z, it follows that $\varphi(\mathrm{T}^k(x)) = y$ for all $k \in \mathbf{N}$. Therefore, by the orbital φ-coherence of T on $\mathcal{O}(x)$ we finally get $z = \mathrm{T}(z)$, i.e., T has a fixed-point in X.

Remark 1.3. If " condition (i) " is replaced by " T is φ-coherent on X " we get a stronger version of Theorem 1.1. Further, the theorem of Furi and Vignoli follows as a consequence of Theorem 1.1 because they require the continuity, the φ-coherence of T and the continuity of φ on all X.

2. Applications of the main theorem

2.1. At first, we want to show how all results (we shall recall some of them!), on the existence of fixed-points for mappings which have a " contractive type behaviour " on the orbits, are consequences of Theorem 1.1.

Let (X, d) be a metric space and let T be a mapping of into itself.

The space X is said to be T-*orbitally complete* if every Cauchy orbital sequence converges in X.

The mapping T is said to be a *generalized contraction* (see Čirič [5]) if

(2.1) for every $x, y \in X$ there exists non-negative numbers $q(x, y)$, $r(x, y)$, $s(x, y)$ and $t(x, y)$ such that

$$\sup_{x \in X} \{q(x, y) + r(x, y) + s(x, y) + \tfrac{1}{2} t(x, y)\} < 1$$

and

$$d(T(x), T(y)) \leq q(x, y) d(x, y) + k(x, y) d(x, T(x)) +$$

$$+ s(x, y) d(y, T(y)) + t(x, y) [d(x, T(y)) + d(y, T(x))]$$

holds,
or equivalently

(2.2) there exists a positive number $q < 1$ such that

$$d(T(x), T(y)) \leq q \max \{d(x, y); d(x, T(x)); d(y, T(y));$$

$$\tfrac{1}{2} [d(x, T(y)) + d(y, T(x))]\}$$

holds for every $x, y \in X$.

LEMMA 2.1. *If* $T: X \to X$ *is a generalized contraction of a* T-*orbitally complete metric space* (X, d) *into itself, then there exists a nice cluster point of* $(T^n(x))$ *for every* $x \in X$.

Proof. Let $x \in X$ and $\mathcal{O}(x)$ be the orbit at x. Assume $T^n(x) \neq T^{n+1}(x)$ for all $n \in \mathbf{N}$ because if $T^{\bar{n}}(x) = T(T^{\bar{n}}(x))$ for some $\bar{n} \in \mathbf{N}$ then $z = T^{\bar{n}}(x)$ is " nice ". Taken $T^{n-1}(x)$ and $T^n(x)$, by (2.2) we get

$$d(T(T^{n-1}(x)), T(T^n(x))) \leq q \max \{d(T^{n-1}(x), T^n(x));$$

$$d(T^{n-1}(x), T(T^{n-1}(x))); d(T^n(x), T^{n+1}(x));$$

$$\tfrac{1}{2} [d(T^{n-1}(x), T(T^n(x))) + d(T^n(x), T(T^{n-1}(x)))]\}.$$

Hence, one of the following relations is satisfied

$$d(\mathrm{T}^n(x), \mathrm{T}^{n+1}(x)) \leq qd(\mathrm{T}^{n-1}(x), \mathrm{T}^n(x)) \leq qd(\mathrm{T}^n(x), \mathrm{T}^{n+1}(x)),$$

since $q < 1$ it is impossible,

$$\leq \frac{q}{2} d(\mathrm{T}^{n-1}(x), \mathrm{T}^{n+1}(x)) \leq q/2 \,[d(\mathrm{T}^{n-1}(x), \mathrm{T}^n(x)) + d(\mathrm{T}^n(x), \mathrm{T}^{n+1}(x))].$$

Therefore,

$$d(\mathrm{T}^n(x), \mathrm{T}^{n+1}(x)) \leq \lambda d(\mathrm{T}^{n-1}(x), \mathrm{T}^n(x)) \qquad 0 \leq \lambda < 1, \tag{2.3}$$

which shows that a generalized contraction is a contraction on the orbits.

By iterating n-times the above process we obtain

$$d(\mathrm{T}^n(x), \mathrm{T}^{n+1}(x)) \leq \lambda^n d(x, \mathrm{T}(x)),$$

and so

$$d(\mathrm{T}^n(x), \mathrm{T}^{n+p}(x)) \leq \frac{\lambda^n}{1-\lambda} d(x, \mathrm{T}(x))$$

for any positive integer p.

But $\lambda < 1$, therefore $(\mathrm{T}^n(x))_{n\in\mathbf{N}}$ is a Cauchy orbital sequence and so, by the T-orbital completeness of T there is a limit point $z \in \mathrm{X}$ of $(\mathrm{T}^n(x))_{n\in\mathbf{N}}$.

It is easy to show now that z is nice (for complete details see the proof of Theorem 2.5 of Čirič [5]).

COROLLARY 2.1 (see Čirič [5], pp. 21-23). *Let* $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ *be a generalized contraction of a* T*-complete metric space* (X, d) *into itself.*

Then T *has a fixed-point in* X.

Proof. Taken $x \in \mathrm{X}$, by Lemma 2.1 there exists a nice cluster point z of $\mathrm{T}^n(x))$. Put $\varphi(y) = d(y, \mathrm{T}(y))$ for any $y \in \mathrm{X}$. It is not hard to see that φ is orbitally continuous at z. Further, by (2.3) of Lemma 2.1, we get

$$\varphi(\mathrm{T}(z)) = d(\mathrm{T}(z), \mathrm{T}^2(z)) \leq \lambda d(z, \mathrm{T}(z)) = \lambda\varphi(z).$$

If $z \neq \mathrm{T}(z)$ then $\varphi(\mathrm{T}(z)) < \varphi(z)$, hence T is orbitally φ-coherent on $\mathcal{O}(x)$. Moreover (2.3) of Lemma 2.1 shows also that $(\varphi(\mathrm{T}^n(x)))_{n\in\mathbf{N}}$ is a nonincreasing sequence of positive real numbers, i.e., it is a convergent sequence.

Remark 2.1. If, in Corollary 2.1, the condition

"T is a generalized contraction"

is replaced by

"there exist a, b, c mappings from $(0, +\infty)$ into $[0, 1)$ with

$$\limsup_{r\to t+} [a(r) + b(r) + c(r)] < 1 \qquad (\text{or} \quad a(t) + b(t) + c(t) < 1)$$

such that

$$d(\mathrm{T}(x), \mathrm{T}(y)) \leq a(d(x, y))\, d(x, y) + b(d(x, y))\, d(x, \mathrm{T}(x)) + \\ + c(d(x, y))\, d(y, \mathrm{T}(y))\text{"}$$

or by

$$\text{“}\, d(\mathrm{T}(x), \mathrm{T}(y)) \leq b\,[d(x, \mathrm{T}(x)) + d(y, \mathrm{T}(y))]\ 0 \leq b < \tfrac{1}{2}\,\text{”}$$

or by

$$\text{“}\, d(\mathrm{T}(x), \mathrm{T}(y)) \leq a\, d(x, y) \quad 0 \leq a < 1\,\text{”}$$

we obtain the theorem of Boyd and Wong ([3], p. 459), Reich's theorem ([12], p. 2), Kannan's theorem ([8], p. 73 and [9], p. 406), Rakotch's theorem ([11], p. 463), and Banach's theorem ([1], p. 106).

2.2. Now we want to investigate mappings satisfying "contractive" conditions weaker than those considered by Sehgal [13] and by Edelstein [6], and to obtain conclusions on the existence of fixed-points.

A mapping T of a metric space (X, d) into itself is said to be *generalized contractive* if

$$d(\mathrm{T}(x), \mathrm{T}(y)) < \max\{d(x, y); d(x, \mathrm{T}(x)); d(y, \mathrm{T}(y)); \tfrac{1}{2}[d(x, \mathrm{T}(y)) + d(y, \mathrm{T}(x))]\}$$

holds for every $x \neq y \in \mathrm{X}$.

THEOREM 2.1. *Let* $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ *be a generalized contractive mapping. If* $z \in \mathrm{X}$ *is a nice cluster point of* $(\mathrm{T}^n(x))$ *for some* $x \in \mathrm{X}$, *then* T *has a fixed-point in* X.

Proof. Because z is a nice cluster point of $(\mathrm{T}^n(x))$ and the metric d is a continuous mapping, it follows that the mapping $\varphi(x) = d(x, \mathrm{T}(x))$ is orbitally continuous at z.

Assume $\mathrm{T}^n(x) \neq \mathrm{T}^{n+1}(x)$ for all $n \in \mathbf{N}$. Because T is a generalized contractive mapping, one of the following relations is satisfied

$$\begin{aligned}\varphi(\mathrm{T}^{n+1}(x)) = d(\mathrm{T}^{n+1}(x), \mathrm{T}(\mathrm{T}^{n+1}(x))) &= d(\mathrm{T}(\mathrm{T}^n(x)), \mathrm{T}(\mathrm{T}^{n+1}(x))) = \\ &< d(\mathrm{T}^n(x), \mathrm{T}^{n+1}(x)) = \varphi(\mathrm{T}^n(x)) \\ &< d(\mathrm{T}^n(x), \mathrm{T}(\mathrm{T}^n(x))) = \varphi(\mathrm{T}^n(x)) \\ &< d(\mathrm{T}^{n+1}(x), \mathrm{T}(\mathrm{T}^{n+1}(x))) = \varphi(\mathrm{T}^{n+1}(x)) \qquad \text{impossible} \\ &< \tfrac{1}{2}[d(\mathrm{T}^n(x), \mathrm{T}(\mathrm{T}^{n+1}(x))) + d(\mathrm{T}^{n+1}(x), \mathrm{T}(\mathrm{T}^n(x)))] \leq \\ &\qquad \leq \tfrac{1}{2}[d(\mathrm{T}^n(x), \mathrm{T}^{n+1}(x)) + d(\mathrm{T}^{n+1}(x), \mathrm{T}(\mathrm{T}^{n+1}(x)))].\end{aligned}$$

Therefore,

$$\varphi(\mathrm{T}^{n+1}(x)) < \varphi(\mathrm{T}^n(x) \qquad \text{for all} \quad n \in \mathbf{N}. \tag{2.4}$$

This implies that T is φ-coherent on X. Moreover $(\varphi(\mathrm{T}^n(x)))_{n \in \mathbf{N}}$ is a decreasing sequence of positive real numbers, it is convergent.

By Theorem 1.1 we get that T has a fixed-point (because T is generalized contractive, this point is unique).

Remark 2.2. If, in Theorem 2.1, the conditions.

" T is generalized contractive and there exists a nice cluster point of $(T^n(x))$, for some $x \in X$ " are replaced by

" T is continuous, there is a subsequence $(T^{n_i}(x))_{i \in \mathbf{N}}$ of $(T^n(x))_{n \in \mathbf{N}}$ which converges and

$$d(T(x), T(y)) < \max\{d(x, y); d(x, T(x)); d(y, T(y))\}$$

whenever

$$x \neq y \in X \text{ "}$$

or by

" T is continuous, there exists a subsequence $(T^{n_i}(x))_{i \in \mathbf{N}}$ of $(T^n(x))_{n \in \mathbf{N}}$ which converges and

$$d(T(x), T(y)) < d(x, y)$$

whenever

$$x \neq y \in X \text{ "}$$

we obtain Sehgal's theorem ([13], p. 573) and Edelstein's theorem ([6], pp. 74-75).

2.3. Now we show that the notion of orbital φ-coherence can be involved by studying existence fixed-point problems for mappings with " diminishing orbital diameters ".

DEFINITION 2.1 (see Kirk [10], p. 107). *For* $x \in X$, *let* $\rho(T^n(x))$ *be the diameter of the orbit* $\mathcal{O}(T^n(x))$. *The mapping* T *is said to have* diminishing orbital diameters *if*

$$\rho(x) < +\infty \qquad \textit{and} \lim_{n \to +\infty} \rho(T^n(x)) = r(x) < \rho(x), \qquad \textit{for all } x \in X$$

such that $\rho(x) > 0$ (*i.e.*, $x \neq T(x)$).

From the definition it follows that $(\rho(T^n(x)))_{n \in \mathbf{N}}$ is a non-increasing bounded from below sequence. Therefore $(\rho(T^n(x)))_{n \in \mathbf{N}}$ converges for all $x \in X$.

THEOREM 2.2. *Let* (X, d) *be a complete metric space and* $T: X \to X$ *have diminishing orbital diameters. If* z *is a nice cluster point of* $(T^n(x))$ *for some* $x \in X$ *and* $\rho(x)$ *is orbitally continuous at* z, *then* $z = T(z)$.

Proof. Put $\rho(x) = \rho(x)$. By the above remark, $(\varphi(T^n(x)))_{n \in \mathbf{N}}$ converges. Evidently T is φ-coherent because if $x \neq T(x)$ then $\lim_{n \to +\infty} \rho(T^n(x)) < \rho(x)$, i.e., there exists $k \in \mathbf{N}$ such that $\rho(T^k(x)) < \rho(x)$, i.e., $\varphi(T^k(x)) < \varphi(x)$. Therefore by Theorem 1.1, z is fixed-point of T.

Remark 2.3. If, in Theorem 2.3, the conditions.

" z is a nice cluster point of $(T^n(x))$, $x \in X$, and $\rho(x)$ is orbitally continuous at z "

are replaced by

(i) "T is continuous, there exists a strong cluster point of $(T^n(x))_{n \in \mathbf{N}}$ and $\rho(x)$ is continuous"

or by

(ii) "T is continuous, there exists a strong cluster point of $(T^n(x))_{n \in \mathbf{N}}$ and for each $x \in X$, the sequence $(T^n(x))_{n \in \mathbf{N}}$ is *stable*, i.e.,

for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$y \in X \, , d(x, y) < \delta \Rightarrow d(T^n(x), T^n(y)) < \varepsilon \qquad \text{for all} \quad n \in \mathbf{N}\text{"}$$

or by

(iii) "$d(T(x), T(y)) \leq d(x, y) \; x, y \in X$"

or by

(iv) "there exists a number $C > 0$, such that $d(T^n(x), T^n(y)) < Cd(x, y)$ holds for all positive numbers n",

we obtain Sehgal's theorem ([13], p. 572), the theorem of Furi and Vignoli ([7], p. 199), the theorem of Belluce and Kirk ([2], p. 142), and Kirk's theorem ([10], p. 110), respectively.

We give an example of a mapping satisfying the hypothesis of Theorem 2.2 for which conditions (i-iv) of Remark 2.3 do not hold.

Example. Let X be the set of the plane defined by

$$X = \{(x, y) : x \geq 0, y \geq 0\}$$

with the Euclidean metric. Let $T : X \to X$ be defined as

$$T(x, y) = \begin{cases} (1, 1) & \text{if} \quad x = y = 0 \\ (x, x^2) & \text{if} \quad (x, y) \in X \, \{(0, 0)\} \end{cases}$$

T is *not* continuous. Each point $(x, x^2) \in X, x \neq 0$ is a fixed-point of T, therefore T is sequentially continuous. Clearly

$$\rho((0, 0)) = \sqrt{2} \qquad \text{and} \quad \rho(T(0, 0)) = 0$$

$$\rho((x, y)) = |y - x^2| \qquad \text{and} \quad \rho(T(x, y)) = 0 \qquad \text{for} \quad x \neq 0$$

$$\rho((0, y)) = \sup\{y, \sqrt{1 + (1 - y)^2}, \sqrt{2}\},$$

$$\rho(T(0, y)) = \rho((0, 0)) = \sqrt{2}, \qquad \text{and} \quad \rho(T^2(0, y)) = 0.$$

That is, T *has* diminishing orbital diameters.

Further,

1) $\rho(x)$ is *not* continuous on X.

In fact, if $(x_n, y_n) \in X, x_n \neq 0$, and $((x_n, y_n))_{n \in \mathbf{N}}$ converges to $(0, 0)$ then $x_n \to 0$ and $y_n \to 0$. Therefore

$$\rho((x_n, y_n)) = |y_n - x_n^2| \to 0 \neq \rho((0, 0)) = 2$$

2) the sequence $(T^n(0, 0))_{n \in \mathbf{N}}$ of iterates at $(0, 0)$ is *not* stable.

In fact, taken $\varepsilon = \sqrt{2}$ for every $\delta > 0$ there exists a point $(0, y) \neq (0, 0)$ such that

$$d((0, 0), (0, y)) < \delta \qquad \text{and} \quad d(T(0, 0), T(0, y)) = \sqrt{2}.$$

3) T does *not* satisfy (iii) and (iv).

Put $p_n = (n, 0)$ and $q_n = (n + 1, 0)$, $n > 0$, we get

$$d(T(p_n), T(q_n)) > (2n + 1) d(p_n, q_n).$$

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