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Submanifolds of real codimension of a complex projective space

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Geometria differenziale. — *Submanifolds of real codimension of a complex projective space.* Nota di MASAFUMI OKUMURA, presentata [(*)] dal Socio B. SEGRE.

RIASSUNTO. — Allo scopo di studiare una sottovarietà reale M di uno spazio proiettivo complesso, si costruisce il sistema di cerchi su M compatibile colla fibrazione di Hopf e che può venire considerato come una sottovarietà di una sfera di dimensione dispari. Così, valendosi della teoria della sommersione, condizioni imposte alla M vengono a tradursi in altre relative ad una sottovarietà di una sfera; e vari esempi al riguardo vengono approfonditi.

INTRODUCTION

It is well known that a $(2n+1)$-dimensional sphere S^{2n+1} is a principal circle bundle over a complex projective space CP^n and that the Riemannian structure on CP^n is given by the submersion $\tilde{\pi}: S^{2n+1} \to CP^n$ [5, 7]. Thus the theory of submersion is one of the most powerful tools for studying a complex projective space and its submanifolds. From this point of view, H. B. Lawson [2] studied real hypersurfaces of a complex projective space and then Y. Maeda [3] and the present author [4] developed this method extensively.

The purpose of the present paper is to establish some relations between a submanifold of CP^n and that of S^{2n+1} which is a principal circle bundle of CP^n. We are mainly concerned with gathering information on the second fundamental tensors of these submanifolds and on the connections of their normal bundles.

In § 1, we state some fundamental formulas for submanifolds of Riemannian manifold and in § 2, we recall fundamental equations of a submersion which are established by B. O'Neill [5], K. Yano and S. Ishihara [7]. Then, in § 3, we consider a submanifold $\overline{M}$ of S^{2n+1} which is a circle bundle over a submanifold M of CP^n. Here we relate fundamental tensors of the submersion $\tilde{\pi}: S^{2n+1} \to CP^n$ and of $\pi: \overline{M} \to M$ as well as the second fundamental tensors of the hypersurfaces $\overline{M}$ and M.

Mean curvature vector fields of M and $\overline{M}$ are discussed in § 4 and a certain pinching theorem is proved in § 5. In § 6 we establish new definition of anti-holomorphic submanifold of a complex manifold and prove some similarities between submanifold of S^{2n+1} and anti-holomorphic submanifold of CP^n.

§ 1. SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD

Let $i: M \to \tilde{M}$ be an isometric immersion of an m-dimensional Riemannian manifold M into $(m+q)$-dimensional Riemannian manifold $\tilde{M}$. The Riemannian metrics g of M and G of $\tilde{M}$ are related by

$$g(X, Y) = G(i(X), i(Y)), \tag{1.1}$$

where X, Y are vector fields on M and we denote also by i the differential of the immersion. The tangent space $T_p(M)$ is identified with a subspace

(*) Nella seduta del 12 aprile 1975.

of $T_{i(p)}(\tilde{M})$. The normal space $N_p(M)$ is the subspace of $T_{i(p)}(\tilde{M})$ consisting of all $X \in T_{i(p)}(\tilde{M})$ which are orthogonal to $T_p(M)$ with respect to the Riemannian metric G. We denote by ∇, and D the Riemannian connection of M and $\tilde{M}$ respectively and by D^N the connection of the normal bundle of M. Let $N_1, \cdots, N_q$ be an orthonormal basis of $N_p(M)$ and extend them to normal vector fields in a neighborhood of p. Then, ∇, D and D^N are related in the following manner:

$$D_{i(X)}\, i(Y) = i(\nabla_X Y) + \sum_{A=1}^{q} G(H_A X, Y) N_A, \tag{1.2}$$

$$D_{i(X)} N_A = -i(H_A X) + D_X^N N_A, \tag{1.3}$$

where H_A is the second fundamental tensor associated with N_A.

We call (1.2) and (1.3) Gauss equation and Weingarten equation respectively. Since $D_X^N N_A$ is normal to M, it is a linear combination of N_A's and so we put

$$D_A^N N_X = \sum_{B=1}^{q} L_{AB}(X) N_B, \tag{1.4}$$

and call L_{AB} the third fundamental tensor of M in $\tilde{M}$. The mean curvature vector N of M is defined by

$$N = \frac{1}{m} \sum_{A=1}^{q} (\text{trace } H_A) N_A, \tag{1.5}$$

and it is well known that N is independent of the choice of N_A's.

Let R, $\tilde{R}$ and R^N be the curvature tensors for ∇, D and D^N respectively. Then we have the following Gauss, and Ricci-Khüne equations:

$$G(\tilde{R}(i(X), i(Y))\, i(Z), i(W)) = g(R(X, Y)Z, W) - \sum_{B=1}^{q} g(H_B Y, Z) g(H_B X, W) + \sum_{B=1}^{q} g(H_B X, Z) g(H_B Y, W), \tag{1.6}$$

$$G(\tilde{R}(i(X), i(Y)) N_A, N_B) = g((H_B H_A - H_A H_B) X, Y) + G(R^N(X, Y) N_A, N_B). \tag{1.7}$$

If the ambient manifold M is a manifold of constant curvature C, it follows that

$$G(R^N(X, Y) N_A, N_B) = g((H_A H_B - H_B H_A) X, Y) \tag{1.8}$$

because the curvature tensor $\tilde{R}$ of $\tilde{M}$ has the form

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = C\{G(\tilde{Y}, \tilde{Z})\tilde{X} - G(\tilde{X}, \tilde{Z})\tilde{Y}\},$$

where X, Y and Z are any vector fields on $\tilde{M}$. Thus for a submanifold M of a manifold of constant curvature the connection of the normal bundle is flat if and only if any H_A and H_B commute.

§ 2. Riemannian submersion

Let $\bar{M}$ and M be differentiable manifolds of dimension $m+1$ and m respectively and assume that there exists a submersion $\pi : \bar{M} \rightarrow M$, that is, assume that π is onto and of maximum rank m everywhere on $\bar{M}$. We further assume that there are given in $\bar{M}$ a vector field $\bar{V}$ which is everywhere tangent to the fibre and a Riemannian metric $\bar{g}$ which satisfies for any $\bar{X}, \bar{Y} \in T_{\bar{p}}(\bar{M})$,

$$\bar{g}(\bar{V}, \bar{V}) = 1, \tag{2.1}$$

$$(L(\bar{V})\bar{g})(\bar{X}, \bar{Y}) = \bar{g}(\bar{\nabla}_{\bar{X}}\bar{V}, \bar{Y}) + \bar{g}(\bar{\nabla}_{\bar{Y}}\bar{V}, \bar{X}) = 0, \tag{2.2}$$

where $L(\bar{V})$ denotes the operator for Lie derivative with respect to $\bar{V}$. Let $\bar{X}$ be a tangent vector at $\bar{p} \in \bar{M}$. Then $\bar{X}$ decomposes as $\bar{X}^V + \bar{X}^H$, where $\bar{X}^V$ is tangent to the fibre through $\bar{p}$ and $\bar{X}^H$ is perpendicular to it. If $\bar{X}=\bar{X}^V$, it is called a *vertical* vector and if $\bar{X} = \bar{X}^H$, it is called *horizontal*.

If a tensor field $\bar{T}$ defined on $\bar{M}$ satisfies $L(\bar{V})\bar{T} = 0$, then it is called an invariant tensor field or a projectable tensor field. Such a tensor field can be regarded as a tensor field defined on M by π.

For any differentiable function f on M define a function f^L on $\bar{M}$ by

$$f^L(\bar{p}) = f(\pi(\bar{p})) = (f \circ \pi)(\bar{p}). \tag{2.3}$$

We call f^L the lift of f. For a vector field X defined on M there exists a unique horizontal vector field X^L on $\bar{M}$ such that for all $\bar{p} \in \bar{M}$ we have

$$\pi X^L_{\bar{p}} = X_{\pi(\bar{p})}, \tag{2.4}$$

and X^L is called the lift of X. We further define the lift u^L of a 1-form u on M by $u^L = \pi^* u$, where π^* denotes the dual map of the differential map of the submersion π. Thus we can define the lift of any type of tensor fields T and S in such a way that

$$(T \otimes S)^L = T^L \otimes S^L, \tag{2.5}$$

where $\otimes$ denotes the operator of the tensor product.

By definition we have easily

$$\pi(X^L) = X, \tag{2.6}$$

$$\pi(\bar{X})^L = \bar{X}^H, \text{ for invariant } \bar{X}. \tag{2.7}$$

Since the Riemannian metric $\bar{g}$ satisfies (2.2), we can define a Riemannian metric g on M by

$$g(X, Y)(p) = \bar{g}(X^L, Y^L)(\bar{p}), \tag{2.8}$$

where $\bar{p}$ is an arbitrary point of $\bar{M}$ such that $\pi(\bar{p}) = p$. Hence we have

$$g(X, Y)^L = \bar{g}(X^L, Y^L). \tag{2.9}$$

The fundamental tensor F of the submersion is a skew-symmetric tensor of type (1.1) on M and is related to covariant differentiation $\bar{\nabla}$ and ∇ in $\bar{M}$ and M, respectively, by the following formulas:

$$\bar{\nabla}_{Y^L} X^L = (\nabla_Y X)^L + \bar{g}(F^L Y^L, X^L)\bar{V} = (\nabla_Y X)^L + g(FY, X)^L \bar{V}, \tag{2.10}$$

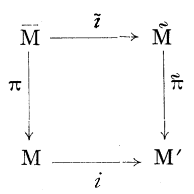
$$\bar{\nabla}_{\bar{V}} X^L = \bar{\nabla}_{X^L} \bar{V} = -F^L X^L. \tag{2.11}$$

This, together with (2.2), implies that

$$\bar{\nabla}_{\bar{V}} \bar{V} = -F^L \bar{V} = 0. \tag{2.12}$$

§ 3. SUBMERSION AND IMMERSION

Let $\tilde{M}$ and M' be differentiable manifolds of dimension $n+p+1$ and $n+p$ respectively and $\tilde{\pi}$ be a submersion $\tilde{\pi}: \tilde{M} \to M'$ which satifies the conditions of § 2. Suppose that $\bar{M}$ is a submanifold of dimension $n+1$ which is immersed in $\tilde{M}$ and respects the submersion $\tilde{\pi}$. That is, suppose that there is a submersion $\pi: \bar{M} \to M$, where M is a submanifold of M' such that the diagram



commutes and the immersion $\bar{i}$ is a diffeomorphism on the fibres.

Let $\bar{V}$ be the unit tangent vector to the fibre of $\bar{M}$ which satisfies (2.2). Then by the commutativity of the diagram we easily see that $\bar{i}(\bar{V})$ is vertical with respect to $\tilde{\pi}$. So we may put

$$\tilde{V} = \bar{i}(\bar{V}). \tag{3.1}$$

Let $\tilde{v}$ be the 1-form on $\tilde{M}$ satisfying

$$\tilde{v}(\tilde{V}) = 1$$

and

$$\tilde{v}(\tilde{X}) = 0$$

for any horizontal vector field $\tilde{X}$ on $\tilde{M}$. The Riemannian metric $\tilde{G}$ of $\tilde{M}$ is given by

$$\tilde{G}(\tilde{X}, \tilde{Y}) = G^L(\tilde{X}, \tilde{Y}) + \tilde{v}(\tilde{X})\tilde{v}(\tilde{Y}), \tag{3.2}$$

from which we know that $G(X', Y') = 0$ implies $\tilde{G}(X'^L, Y'^L) = 0$.

We denote by $\bar{g}$ the induced Riemannian metric of $\bar{M}$. Then for a vector field X on M, we have

$$\tilde{G}(\bar{i}(X^L), \tilde{V}) = \tilde{G}(\bar{i}(X^L), \bar{i}(\bar{V})) = \bar{g}(X^L, \bar{V}) = 0,$$

which shows that $\tilde{i}(X^L)$ is horizontal. On the other hand from the commutativity of the diagram we know that if $\bar{X}$ is an invariant vector field on $\bar{M}$, $\tilde{i}(\bar{X})$ is also an invariant vector field on $\tilde{M}$. Hence we have

$$\tilde{\pi}(\tilde{i}(X^L)) = i(\pi(X^L)) = i(X),$$

which, together with (2.7) implies that

$$\tilde{i}(X^L) = \tilde{i}(X^L)^H = \tilde{\pi}(\tilde{i}(X^L))^L = i(X)^L. \tag{3.3}$$

Let $N_A (A = 1, 2, \cdots, p)$ be normal vector fields to M which are mutually orthonormal at a point $x \in M$ and put $\bar{N}_A = N_A^L$. Then $\bar{N}_A$'s are also normal vector fields to $\bar{M}$ which are mutually orthonormal at any point $y \in \bar{M}$ satisfying $\pi(y) = x$. In fact, by (3.2), it follows that

$$\bar{G}(\bar{N}_A, \tilde{i}(X^L)) = \bar{G}(\bar{N}_A, i(X)^L) = G^L(N_A^L, i(X)^L) +$$
$$+ \tilde{v}(N_A^L)\,\tilde{v}(i(X)^L) = G(N_A, i(X))^L = 0,$$

$$\bar{G}(\bar{N}_A, \bar{N}_B) = \bar{G}(N_A^L, N_B^L) = G^L(N_A^L, N_B^L) + \tilde{v}(N_A^L)\,\tilde{v}(N_B^L) =$$
$$= G(N_A, N_B)^L = \delta_{AB}.$$

Let $\bar{D}$, $\bar{\nabla}$, D and ∇ be respectively the Riemannian connections of $\tilde{M}$, $\bar{M}$, M′ and M. By means of the Gauss equation for submanifold, we have

$$\bar{D}_{\tilde{i}(X^L)}\,\tilde{i}(Y^L) = \tilde{i}(\bar{\nabla}_{X^L} Y^L) + \Sigma_{A=1}^{p}\,\bar{g}(\bar{H}_A X^L, Y^L)\,N_A^L =$$
$$= \tilde{i}((\nabla_X Y)^L + \bar{g}(F^L X^L, Y^L)\,\bar{V}) + \Sigma_{A=1}^{p}\,\bar{g}(\bar{H}_A X^L, Y^L)\,N_A^L,$$

from which

$$(D_{i(X)}\,i(Y))^L + \bar{G}('F^L\,i(X)^L, i(Y)^L)\,\tilde{V} = \tilde{i}(\nabla_X Y)^L +$$
$$+ \bar{g}(F^L X^L, Y^L)\,\tilde{i}(\bar{V}) + \Sigma_{A=1}^{p}\,\bar{g}(\bar{H}_A X^L, Y^L)\,N_A^L.$$

Comparing the vertical parts and horizontal parts, we have

$$\bar{G}('F^L\,i(X)^L, i(Y)^L) = \bar{g}(F^L X^L, Y^L), \tag{3.4}$$
$$(D_{i(X)}\,i(Y))^L = \tilde{i}(\nabla_X Y)^L + \Sigma_{A=1}^{p}\,\bar{g}(\bar{H}_A X^L, Y^L)\,N_A^L.$$

Using the Gauss equation again, we get

$$\bar{g}(\bar{H}_A X^L, Y^L) = g(H_A X, Y)^L. \tag{3.5}$$

From (2.9) and (3.4), we have also

$$G('Fi(X), i(Y)) = g(FX, Y). \tag{3.6}$$

Next we consider the transforms $'Fi(X)$ and $'FN_A$ of $i(X)$ and N_A by the fundamental tensor $'F$ of the submersion $\tilde{\pi}$. By means of (3.6) they can be written as

$$'Fi(X) = i(FX) + \Sigma_{A=1}^{p} u_A(X) N_A , \tag{3.7}$$

$$'FN_A = -i(U_A) + \Sigma_{B=1}^{p} \lambda_{AB} N_B , \tag{3.8}$$

and we easily see that

$$g(U_A, X) = u_A(X) . \tag{3.9}$$

We denote by D^N and $\bar{D}^N$ the connections of the normal bundle of M in M' and $\bar{M}$ in $\tilde{M}$ respectively. By definition of $\bar{D}^N$, we have

$$\bar{D}^N_{X^L} N^L_A = \bar{D}_{\tilde{i}(X^L)} N^L_A + \tilde{i}(\bar{H}_A X^L),$$

from which

$$\begin{aligned} \bar{D}^N_{X^L} N^L_A &= (D_{i(X)} N_A)^L + \bar{G}('F^L i(X)^L, N^L_A) \tilde{V} + \tilde{i}(\bar{H}_A X^L) \\ &= -i(H_A X)^L + (D^N_X N_A)^L + G('Fi(X), N_A)^L \tilde{V} + \tilde{i}(\bar{H}_A X^L). \end{aligned}$$

Comparing the horizontal parts and vertical parts and using (3.1), we get

$$\bar{D}^N_X N^L_A = (D^N_X N_A)^L, \tag{3.10}$$

$$g(U_A, X)^L = G('Fi(X), N_A)^L = -\bar{g}(\bar{H}_A X^L, \bar{V}), \tag{3.11}$$

because of (3.7) and (3.9). The normal connection being expressed by the third fundamental tensor L_{AB} as (1.4), (3.10) is nothing but

$$\bar{L}_{AB}(X^L) = L_{AB}(X)^L. \tag{3.12}$$

Consider the covariant differentiation of N^L_A in the direction of $\tilde{V}$. By (1.2) and (3.1), it follows that

$$\bar{D}_{\tilde{i}(\bar{V})} N^L_A = -\tilde{i}(\bar{H}_A \bar{V}) + \bar{D}^N_{\bar{V}} N^L_A = -\tilde{i}(\bar{H}_A \bar{V}) + \Sigma_{B=1}^{p} \bar{L}_{AB}(\bar{V}) N^L_B .$$

Substituting (2.11) into the above equation, we have

$$-'F^L N^L_A = -\tilde{i}(\bar{H}_A \bar{V}) + \Sigma_{B=1}^{p} \bar{L}_{AB}(\bar{V}) N^L_B ,$$

from which

$$\lambda^L_{AB} = \bar{G}('F^L N^L_A , N^L_B) = -\bar{L}_{AB}(\bar{V}), \tag{3.13}$$

because of (3.8).

§ 4. MEAN CURVATURE VECTOR FIELDS

In this section we want to relate the conditions imposed on the mean curvature vectors of M and $\bar{M}$. First of all we prove the

LEMMA 4.1. *For any point $\bar{p} \in \bar{M}$, we have*

$$(\text{trace } \bar{H}_A)(\bar{p}) = (\text{trace } H_A)(\pi(\bar{p})) = (\text{trace } H_A)^L(\bar{p}). \tag{4.1}$$

Proof. Let $\{E_1, \cdots, E_n\}$ be an orthonormal basis at $T_{\pi(\bar{p})}(M)$ and choose an orthonormal basis $\{\bar{E}_1, \cdots, \bar{E}_{n+1}\}$ at $T_{\bar{p}}(\bar{M})$ in such a way that $\bar{E}_i = E_i^L$ for $i = 1, \cdots, n$ and $\bar{E}_{n+1} = \bar{V}$. Then we get

$$\begin{aligned}\text{trace } \bar{H}_A &= \Sigma_{\alpha=1}^{n+1} \bar{g}(\bar{H}_A \bar{E}_\alpha, \bar{E}_\alpha) = \Sigma_{i=1}^{n} \bar{g}(\bar{H}_A E_i^L, E_i^L) + \bar{g}(\bar{H}_A \bar{V}, \bar{V}) \\ &= \Sigma_{i=1}^{n} g(H_A E_i, E_i)^L + \bar{g}(\bar{H}_A \bar{V}, \bar{V}) = (\text{trace } H_A)^L + \bar{g}(\bar{H}_A \bar{V}, \bar{V}),\end{aligned}$$

because of (3.5). On the other hand, from (1.2), we have

$$\bar{D}_{\bar{V}}^{\tilde{}} \tilde{V} = \bar{D}_{\tilde{i}(\bar{V})}\, i(\bar{V}) = i(\bar{\nabla}_{\bar{V}} \bar{V}) + \Sigma_{A=1}^{p} \bar{g}(\bar{H}_A \bar{V}, \bar{V}) N_A = 0,$$

which, together with (2.12), implies that

$$\bar{g}(\bar{H}_A \bar{V}, \bar{V}) = 0, \qquad A = 1, 2, \cdots, p. \tag{4.2}$$

Thus we have (4.1). This completes the proof.

Let N and $\bar{N}$ be the mean curvature vector field of M and $\bar{M}$ respectively. Then, by Lemma 4.1, it follows that

$$\bar{N} = \frac{1}{n+1} \Sigma_{A=1}^{p} (\text{trace } \bar{H}_A) \bar{N}_A = \frac{1}{n+1} \Sigma_{A=1}^{p} (\text{trace } H_A)^L N_A^L = \frac{n}{n+1} N^L. \tag{4.3}$$

LEMMA 4.2 *If the mean curvature vector field* $\bar{N}$ *of* $\bar{M}$ *is parallel with respect to the induced connection of the normal bundle so is the mean curvature vector field* N *of* M.

Proof. Letting $\bar{D}_{X^L}^N$ act on $\bar{N}$, we get

$$\begin{aligned}(n+1)\bar{D}_{X^L}^N \bar{N} &= \Sigma_{A=1}^{p} \{X^L (\text{trace } \bar{H}_A) \bar{N}_A + (\text{trace } \bar{H}_A) \bar{D}_{X^L}^N \bar{N}_A\} \\ &= \Sigma_{A=1}^{p} \{X^L (\text{trace } H_A)^L N_A^L + (\text{trace } H_A)^L (D_X^N N_A)^L\} \\ &= \Sigma_{A=1}^{p} \{X (\text{trace } H_A) N_A + (\text{trace } H_A) D_X^N N_A\}^L \\ &= n (D_X^N N)^L,\end{aligned} \tag{4.4}$$

because of (3.10). Thus $\bar{D}_{X^L}^N \bar{N} = 0$ implies that $D_X^N N = 0$. This completes the proof.

Next we relate the length of the second fundamental tensors of M and $\bar{M}$. From (3.5) and

$$g(H_A X, Y)^L = g(H_A X, Y) \circ \pi = \bar{g}((H_A X)^L, Y^L),$$

we obtain

$$\bar{H}_A X^L = (H_A X)^L + \bar{g}(\bar{H}_A X^L, \bar{V}) \bar{V}. \tag{4.5}$$

We choose an orthonormal basis $\bar{E}_\alpha$ such as the one we have chosen in the proof of Lemma 4.1, and we have

$$\begin{aligned}\text{trace } \bar{H}_A^2 &= \Sigma_{\alpha=1}^{n+1} \bar{g}(\bar{H}_A^2 \bar{E}_\alpha, \bar{E}_\alpha) = \Sigma_{i=1}^{n} \bar{g}(\bar{H}_A^2 E_i^L, E_i^L) + \bar{g}(\bar{H}^2 \bar{V}, \bar{V}) \\ &= \sum_{i=1}^{n} \bar{g}(\bar{H}_A((H_A E_i)^L + \bar{g}(\bar{H}_A E_i^L, \bar{V}) \bar{V}), E_i^L) + \bar{g}(\bar{H}_A \bar{V}, \bar{H}_A \bar{V}) \\ &= \sum_{i=1}^{n} \bar{g}(\bar{H}_A (H_A E_i)^L, E_i^L) + \sum_{i=1}^{n} \bar{g}(\bar{H}_A E_i^L, \bar{V}) \bar{g}(\bar{H}_A \bar{V}, E_i^L) + \bar{g}(\bar{H}_A \bar{V}, \bar{H}_A \bar{V}).\end{aligned}$$

Substituting (3.11) into the last equation and making use of the fact that

$$\bar{g}(\bar{H}_A \bar{V}, \bar{H}_A \bar{V}) = \sum_{\alpha=1}^{n+1} \bar{g}(\bar{H}_A \bar{V}, \bar{E}_\alpha)\, \bar{g}(\bar{H}_A \bar{V}, \bar{E}_\alpha) = \sum_{i=1}^{n} \bar{g}(\bar{H}_A \bar{V}, E_i^L)\, \bar{g}(\bar{H}_A \bar{V}, E_i^L),$$

we obtain

$$\text{trace } \bar{H}_A^2 = \sum_{i=1}^{n} \{g(H_A^2 E_i, E_i)^L + 2g(E_i, U_A)^L g(E_i, U_A)^L\} = (\text{tracce } H_A^2)^L + 2g(U_A, U_A)^L,$$

because of (4.2). Hence we have

$$\sum_{A=1}^{p} \text{trace } \bar{H}_A^2 = \left(\sum_{A=1}^{p} \text{trace } H_A^2\right)^L + 2\sum_{A=1}^{p} g(U_A, U_A)^L. \tag{4.6}$$

THEOREM 4.1. $\sum_{A=1}^{p}$ *trace* $\bar{H}_A^2 \geq \left(\sum_{A=1}^{p} \textit{trace } H_A^2\right)^L$ *is always valid. The equality holds if, and only if, the submanifold* M *is invariant under* 'F.

If $\tilde{i}$ is a totally geodesic immersion, from (4.6) we have

THEOREM 4.2 *Let* $\tilde{i}$ *be a totally geodesic immersion of a Riemannian manifold* $\bar{M}$ *in* $\tilde{M}$ *which respects the submersion* $\tilde{\pi} : \tilde{M} \to M'$*; then i is also totally geodesic and the tangent space of* M *is invariant under* 'F.

§ 5. REAL SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACES

Let S^{n+p+1} be an odd-dimensional unit sphere in an $(n+p+2)$-dimensional Euclidean space $E^{n+p+2} = C^{(n+p+2)/2}$ and $\tilde{J}$ the natural almost complex structure on $C^{(n+p+2)/2}$. The image $\tilde{V} = \tilde{J}\tilde{N}$ of the outward unit normal vector $\tilde{N}$ to S^{n+p+1} by the almost complex structure defines a unit tangent vector field on S^{n+p+1} and the integral curves of $\tilde{V}$ are great circles S^1 in S^{n+p+1} which are fibres of the standard fibration $\tilde{\pi}$,

$$S^1 \to S^{n+p+1} \xrightarrow{\tilde{\pi}} CP^{(n+p)/2} \tag{5.1}$$

onto complex projective space. The usual Riemannian structure on $CP^{(n+p)/2}$ is characterized by the fact that $\tilde{\pi}$ is a submersion.

Let M^n be a submanifold of real codimension p of a complex projective space $CP^{(n+p)/2}$. Then the principal circle bundle $\bar{M}^{n+1}$ over M^n is a submanifold of codimension p of S^{n+p+1} and the natural immersion $\bar{M}^{n+1}$ into S^{n+p+1} respects the submersion $\tilde{\pi}$. Thus S^{n+p+1} and $CP^{(n+p)/2}$ are in the same situation as $\tilde{M}$ and M' respectively, so we continue to use the same notations as in the preceding sections.

In S^{n+p+1} we have the family of products

$$M_{q,r} = S^q \times S^r$$

where $q + r = n + 1$. By choosing the spheres to lie in complex subspaces, we get fibrations $S^1 \to M_{2q+1,2r+1} \to M_{q,r}^c$, which are compatible with (5.1) where $q + r = (n-1)/2$. The almost complex structure J of $CP^{(n+p)/2}$ is nothing but the fundamental tensor of the submersion $\tilde{\pi}$, that is,

$$J^L \tilde{X} = -\bar{D}_{\tilde{X}} \tilde{V} \quad , \quad \tilde{X} \in T(S^{n+p+1}), \tag{5.2}$$

and the curvature tensor of the complex projective space is given by

$$(5.3)\qquad R'(X', Y')Z' = G(Y', Z')X' - G(X', Z')Y' + G(JY', Z')JX' - \\ - G(JX', Z')JY' - 2G(JY', Y')JZ'.$$

which, together with (3.8), implies that

$$(5.4)\qquad G(R'(i(X), i(Y))N_A, N_B) = g(U_A, Y)g(U_B, X) - \\ - g(U_A, X)g(U_B, Y) - 2g(FX, Y)\lambda_{AB}.$$

Combining this equation with (1.7), we have

$$(5.5)\qquad G(R^N(X, Y)N_A, N_B) = g([H_A, H_B]X, Y) + g(U_A, Y)g(U_B, X) - \\ - g(U_A, X)g(U_B, Y) - 2g(FX, Y)\lambda_{AB}.$$

On the other hand, from (3.5) and (4.5), it follows that

$$g(H_A H_B X, Y)^L = \bar{g}(\bar{H}_A (H_B X)^L, Y^L) = \bar{g}(\bar{H}_A \bar{H}_B X^L, Y^L) - \\ - \bar{g}(\bar{H}_B X^L, \bar{V})\,\bar{g}(\bar{H}_A \bar{V}, Y^L),$$

from which, together with (3.11), we get

$$(5.6)\qquad g([H_A, H_B]X, Y)^L = \bar{g}([\bar{H}_A, \bar{H}_B]X^L, Y^L) - \\ - g(U_B, X)^L g(U_A, Y)^L + g(U_B, Y)^L g(U_A, X)^L.$$

If the normal bundle of $\bar{M}$ of S^{n+p+1} is flat, then by (1.8),

$$\bar{g}([\bar{H}_A, \bar{H}_B]X^L, Y^L) = 0,$$

and so

$$(5.7)\qquad g([H_A, H_B]X, Y) = -g(U_B, X)g(U_A, Y) + g(U_B, Y)g(U_A, X).$$

Substituting (5.7) into (5.5), we have

$$(5.8)\qquad G(R^N(X, Y)N_A, N_B) = -2g(FX, Y)\lambda_{AB}.$$

Thus we have proved

LEMMA 5.1 *If in a submanifold* $\bar{M}$ *of an odd-dimensional sphere* S^{n+p+1} *the connection of the normal bundle is flat, we have* (5.8).

For totally geodesic submanifolds of a complex projective space, we have

THEOREM 5.1. *A compact, totally geodesic submanifold of real codimension* $p < (n+3)/4$ *of a complex projective space* $CP^{(n+p)/2}$ *is necessarily a complex submanifold and consequently a complex projective space* $CP^{n/2}$.

Proof. Since G is the Hermitian metric of $CP^{(n+p)/2}$, it follows that

$$1 = G(JN_A , JN_A) = G(i(U_A), i(U_A)) + G\left(\sum_{B=1}^{p} \lambda_{AB} N_B , \sum_{C=1}^{p} \lambda_{AC} N_C\right) =$$
$$= g(U_A , U_A) + \sum_{B} \lambda_{AB} \lambda_{AB}$$

and then

$$\sum_{A=1}^{p} g(U_A , U_A) = p - \sum_{A,B} \lambda_{AB} \lambda_{AB} \leqq p. \tag{5.9}$$

Thus, combining this with (4.6), we get

$$\sum_{A=1}^{p} \operatorname{trace} \bar{H}_A^2 = 2 \sum_{A=1}^{p} g(U_A , U_A) \leqq 2p < \frac{n+1}{2-1/p} ,$$

because of $p < \frac{n+3}{4}$. Applying Simons' result [6], we obtain that $\bar{M}$ is totally geodesic. By virtue of Theorem 4.2, M is a complex submanifold and consequently a complex projective space $CP^{n/2}$.

COROLLARY. *There is no odd-dimensional, compact totally geodesic submanifold of codimension* $p < \frac{n+3}{4}$ *of a complex projective space.*

THEOREM 5.2. *If a compact minimal submanifold* M *of real codimension* p *of a complex projective space* $CP^{(n+p)/2}$ *satisfies*

$$\sum_{A=1}^{p} \text{traee } H_A^2 < \frac{n+3-4p}{2-1/p} , \tag{5.10}$$

M *is a totally geodesic complex projective space* $CP^{n/2}$.

Proof. We note that (5.9) is still valid for any submanifold M. Combining (4.6) and (5.9), we have

$$\sum_{A=1}^{p} \operatorname{trace} \bar{H}_A^2 \leq \sum_{A=1}^{p} (\operatorname{trace} \bar{H}_A^2)^L + 2p < \frac{n+3-4p}{2-1/p} + 2p = \frac{n+1}{2-1/p} . \tag{5.11}$$

On the other hand Lemma 4.1 shows that if M is minimal, $\bar{M}$ is also minimal. Thus applying Simons' result to (5.11), we obtain that $\bar{M}$ is totally geodesic. Thus Theorem 4.2 shows that M is totally geodesic $CP^{n/2}$.

§ 6. ANTI-HOLOMORPHIC SUBMANIFOLDS

As is well known, a complex submanifold (holomorphic submanifold) of a complex manifold is characterized by the fact that at any point of the submanifold M the tangent space is invariant under the action of the almost complex structure J of the ambient manifold, that is, for any $p \in M$, $T_p(M) = J(T_p(M))$. Since $J^2 = -$ identically, this condition is equivalent to the fact that, at any point of M, the normal space is invariant under J; that is, $N_p(M) = J(N_p(M))$. Now we consider such a subamnifold of a complex manifold that at any point of the submanifold we have

$$JN_p(M) \cap N_p(M) = \{0\}. \tag{6.1}$$

The author calls this submanifold an *anti-holomorphic submanifold*. It should be remarked that some authors call anti-holomorphic a submanifold that satisfies $JT_p(M) \cap T_p(M) = \{0\}$. But it seems to the author that our new definition is preferable being less exacting than the old definition; for example, any real hypersurface of a complex manifold is anti-holomorphic in our sense.

In this section we show that some conditions in $\bar{M}$ of S^{n+p+1} are naturally inherited by anti-holomorphic submanifolds of M of $CP^{(n+p)/2}$.

PROPOSITION 6.1 *Let* M *be an n-dimensional anti-holomorphic submanifold of a complex projective space* $CP^{(n+p)/2}$ *of real codimension p and* $\pi : \bar{M} \to M$ *the submersion which is compatible with the submersion* $\tilde{\pi} : S^{n+p+1} \to CP^{(n+p)/2}$. *Then the mean curvature vector field* N *of* M *is parallel with respect to the induced connection of the normal bundle if, and only, so is* $\bar{N}$ *of* $\bar{M}$.

Proof. By definition of mean curvature vector field, it follows that

$$\bar{D}^N_{\bar{V}} \bar{N} = \sum_{A=1}^{p} (\bar{V} (\text{trace } \bar{H}_A) \bar{N}_A + (\text{trace } \bar{H}_A) \bar{D}^N_{\bar{V}} N_A).$$

Since Lemma 4.1 shows that trace $\bar{H}_A$ is an invariant function with respect to $\bar{V}$ the first term of the right hand side of the last equation vanishes. Moreover, by (1.4), (3.8) and (3.13), we get

$$\bar{D}^N_{\bar{V}} N_A = \sum_{B=1}^{p} \bar{L}_{AB} (\bar{V}) N^L_B = -\sum_{B=1}^{p} \lambda_{AB} N^L_B = 0. \tag{6.2}$$

Combining (4.4) and (6.2), we know that $\bar{N}$ is parallel with respect to the connection of the normal bundle. Conversely if $\bar{N}$ is parallel, Lemma 4.2 shows that so is N. This completes the proof.

From (3.10), we easily prove

PROPOSITION 6.2. *Let* $\bar{M}$ *be a submanifold of* S^{n+p+1} *whose connection induced to the normal bundle is flat and* $\bar{M}$ *agrees with the submersion* $\tilde{\pi} : S^{n+p+1} \to CP^{(n+p)/2}$. *Then the induced connection of the normal bundle of the base submanifold* M *of* $CP^{(n+p)/2}$ *is flat if, and only if,* M *is anti–holomorphic.*

We prove next the

THEOREM 6.1. *Let* M *be an n-dimensional, compact, minimal, anti–holomorphic submanifold of a complex projective space* $CP^{(n+p)/2}$. *If, everywhere on* M, *we have*

$$\sum_{A=1}^{p} \textit{trace } H_A^2 \leq \frac{n+3-4p}{2-1/p}, \tag{6.3}$$

then M *is* $M^c_{q,r}$ *in* $CP^{(n+1)/2}$.

Proof. Since M is anti-holomorphic, we have

$$\sum_{A=1}^{p} g(U_A, U_A) = p, \tag{6.4}$$

because of (3.8) and (5.9). Thus from (4.6), we get

$$\sum_{A=1}^{p} \operatorname{trace} \bar{H}_A^2 = \sum_{A=1}^{p} (\operatorname{trace} H_A^2)^L + 2p \leqq \frac{n+1}{2-1/p} \cdot \tag{6.5}$$

If the equality is not satisfied in (6.5), we see that $\bar{M}$ is a great sphere of S^{n+p+1} and consequently M is a complex projective space. But, M being anti-holomorphic, this is impossible. Thus the equality must be satisfied.

Making use of the Chern-do Carmo-Kobayashi's result [1], we know that $\bar{M}$ is isometric with $S^m(\sqrt{m/(n+1)}) \times S^{n-m+1}(\sqrt{n-m+1)/(n+1)})$ in S^{n+1}. Since $\bar{M}$ is compatible with the submersion $\tilde{\pi}$, m must be an odd number, say $m = 2q+1$. Hence $M = M_{q,r}^c$. This completes the proof.

As a special occurrence, we consider the case $p = 1$. Then we have

COROLLARY [2]. *Let* M *be a compact, real minimal hypersurface of* $CP^{(n+1)/2}$ *on which the inequality*

$$\operatorname{trace} H^2 \leqq n-1\,, \tag{6.6}$$

holds. Then trace $H^2 = n-1$ *and* M *is isometric with* $M_{q,r}^c$.

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