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Picone's Identity for Ordinary Differential Operators of Even Order

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Equazioni differenziali ordinarie. — *Picone's Identity for Ordinary Differential Operators of Even Order.* Nota di TAKAŜI KUSANO e NORIO YOSHIDA, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — In questo lavoro la ben nota identità di M. Picone è generalizzata agli operatori differenziali ordinari autoaggiunti di ordine superiore. Tale identità generalizzata è impiegata per conseguire teoremi di confronto del tipo di Sturm e criteri di non-oscillazione per le soluzioni di equazioni (o diseguaglianze) relative a tali operatori.

1. INTRODUCTION

The classical Picone identity is a fundamental tool in the qualitative study of the Sturm-Liouville equations; see for example Sansone [11]. It has received wide attention and has been the object of various generalizations; we refer, in particular, to Cimmino [1, 2], Eatham [5], Halanay and Šandor [6], Kreith [7, 8] and Kuks [9] for generalizations to higher order ordinary differential equations and systems.

It is the purpose of this paper to establish Picone-type identities for (both scalar and matrix) ordinary differential operators of arbitrary even order and to present their applications to qualitative properties of solutions of the associated differential equations (or inequalities) and systems. We establish generalized Picone identities in Section 2, prove Sturmian comparison theorems in Section 3, and then develop disconjugacy criteria in Section 4.

2. PICONE-TYPE IDENTITIES

A) We shall first consider the scalar ordinary differential operators

$$lu \equiv \sum_{j=0}^{n} (-1)^j (a_j u^{(j)})^{(j)}, \tag{1}$$

$$mw \equiv \sum_{j=0}^{n} (-1)^j (b_j w^{(j)})^{(j)}, \tag{2}$$

where a_j and b_j are real-valued functions of class C^j in an interval I, $j = 0, 1, \cdots, n$. The domains $D_l(I)$ and $D_m(I)$ of l and m, respectively, are defined to be the sets of all real-valued functions of class C^{2n} in I.

(*) Nella seduta del 12 aprile 1975.

THEOREM 1 A. *If* $u \in D_l(I)$, $w \in D_m(I)$ *and if none of* $w, w', \cdots, w^{(n-1)}$ *vanish in* I, *then*

$$(3)\qquad \frac{d}{dx}\sum_{k=0}^{n-1}(-1)^k u^{(k)}\left[\frac{u^{(k)}}{w^{(k)}}\sum_{j=k+1}^{n}(-1)^j(b_j w^{(j)})^{(j-k-1)}-\right.$$
$$\left.-\sum_{j=k+1}^{n}(-1)^j(a_j u^{(j)})^{(j-k-1)}\right]=\frac{u}{w}(umw-wlu)+\sum_{j=0}^{n}(a_j-b_j)(u^{(j)})^2$$
$$+\sum_{k=1}^{n}\left(\frac{u^{(k)}}{w^{(k)}}-\frac{u^{(k-1)}}{w^{(k-1)}}\right)^2(-1)^k w^{(k)}\sum_{j=k}^{n}(-1)^j(b_j w^{(j)})^{(j-k)}.$$

The identity (3) can be readily derived by a straightforward calculation. The verification is left to the reader.

B) We shall next consider the vector and matrix differential operators

$$(4)\qquad Lv \equiv \sum_{j=0}^{n}(-1)^j(A_j v^{(j)})^{(j)},$$

$$(5)\qquad MW \equiv \sum_{j=0}^{n}(-1)^j(B_j W^{(j)})^{(j)},$$

where A_j and B_j are real symmetric $N\times N$ matrix functions of class $C^j(I)$, $j=0, 1, \cdots, n$. The domain $D_L(I)$ of L is the set of all real $N\times 1$ column vector functions v of class $C^{2n}(I)$, while the domain $D_M(I)$ of M is the set of all real $N\times N$ matrix functions W of class $C^{2n}(I)$.

Motivated by Morse [10] and Kuks [9], we say that an $N\times N$ matrix function W is *prepared with respect to* M (or simply M-*prepared*) if

$$(6)\qquad \begin{aligned}(W^{(k-1)})^T\sum_{j=k}^{n}(-1)^j(B_j W^{(j)})^{(j-k)} &= \left(\sum_{j=k}^{n}(-1)^j(B_j W^{(j)})^{(j-k)}\right)^T W^{(k-1)},\\ (W^{(k)})^T\sum_{j=k}^{n}(-1)^j(B_j W^{(j)})^{(j-k)} &= \left(\sum_{j=k}^{n}(-1)^j(B_j W^{(j)})^{(j-k)}\right)^T W^{(k)},\end{aligned}$$

for $k=1, \cdots, n$, where the superscript T denotes taking the transpose of a matrix under consideration.

We remark that in the simplest case when the operator M is generated by a scalar operator m of the form (2), that is, when $B_j = b_j E_N$, E_N being the $N\times N$ identity matrix, every diagonal matrix W of the form $W = wE_N$ is automatically prepared with respect to M.

THEOREM 1 B. *Let* $v \in D_L(I)$, $W \in D_M(I)$ *and suppose* W *is prepared with respect to* M. *If none of* $W, W', \cdots, W^{(n-1)}$ *are singular in* I, *then*

$$(7)\qquad \frac{d}{dx}\sum_{k=0}^{n-1}(-1)^k(v^{(k)})^T\left[\left(\sum_{j=k+1}^{n}(-1)^j(B_j W^{(j)})^{(j-k-1)}\right)(W^{(k)})^{-1}v^{(k)}\right.$$
$$\left.-\sum_{j=k+1}^{n}(-1)^j(A_j v^{(j)})^{(j-k-1)}\right]$$

$$= v^{T}(MW \cdot W^{-1} v - Lv) + \sum_{j=0}^{n} (v^{(j)})^{T} (A_j - B_j) v^{(j)}$$

$$+ \sum_{k=1}^{n} [(W^{(k)})^{-1} v^{(k)} - (W^{(k-1)})^{-1} v^{(k-1)}]^{T} \cdot$$

$$\cdot \left[(-1)^{k} (W^{(k)})^{T} \sum_{j=k}^{n} (-1)^{j} (B_j W^{(j)})^{(j-k)} \right] \cdot$$

$$\cdot [(W^{(k)})^{-1} v^{(k)} - (W^{(k-1)})^{-1} v^{(k-1)}].$$

Since the derivation of (7) is also straightforward, we omit the proof. We note that (7) reduces to (3) in the scalar case when $N = 1$.

Remark. Theorems 1 A and 1 B generalize the identities obtained by Kreith [7] and Kuks [9], respectively, for the fourth order case.

3. Sturmian comparison theorems

A) Using the identity (3) we can prove the following comparison theorem which is a generalization of a result of Kreith [7].

Theorem 2 A. *Suppose there exists a nontrivial real-valued function* $u \in D_l [x_1, x_2]$ *which satisfies*

$$\int_{x_1}^{x_2} ulu \, dx \leqq 0,$$

$$u(x_i) = u'(x_i) = \cdots = u^{(n-1)}(x_i) = 0, \qquad i = 1, 2,$$

$$\int_{x_1}^{x_2} \sum_{j=0}^{n} (a_j - b_j)(u^{(j)})^2 \, dx \geqq 0.$$

If $w \in D_m [x_1, x_2]$ *satisfies*

$wmw \geqq 0$ in (x_1, x_2), *where* $b_n \geqq 0$,

$$(-1)^{k} w^{(k)} \sum_{j=k}^{n} (-1)^{j} (b_j w^{(j)})^{(j-k)} \geqq 0 \quad in \quad (x_1, x_2), \quad k = 1, \cdots, n-1,$$

$$\sum_{j=p}^{n} (-1)^{j} (b_j w^{(j)})^{(j-p)} \neq 0 \quad in \quad (x_1, x_2) \quad for\ some \quad p \in \{1, \cdots, n-1\},$$

then at least one of $w, w', \cdots, w^{(n-1)}$ *has a zero in* (x_1, x_2) *unless* w *is a constant multiple of* u *in the case* $n = 1$.

Proof. Suppose none of $w, w', \cdots, w^{(n-1)}$ vanish in (x_1, x_2). In this case it should be noticed that at least one of $w, w', \cdots, w^{(n-1)}$ must vanish at either

$x = x_1$ or $x = x_2$, since otherwise the Picone identity (3) together with the above hypotheses would imply

$$Q[u, w] \equiv \int_{x_1}^{x_2} \left(\frac{u^{(p)}}{w^{(p)}} - \frac{u^{(p-1)}}{w^{(p-1)}} \right)^2 (-1)^p w^{(p)} \sum_{j=p}^{n} (-1)^j (b_j w^{(j)})^{(j-p)} \, dx = 0,$$

which would yield a contradiction that $u^{(p-1)} = cw^{(p-1)}$ in $[x_1, x_2]$ for some nonzero constant c.

Following Dunninger [4], we now take a sequence $\{u_\nu\}$ of C^∞ functions having compact support in (x_1, x_2) and converging to u as $\nu \to \infty$ in the norm $\|\cdot\|$ defined by

$$\|\varphi\|^2 \equiv \int_{x_1}^{x_2} \sum_{j=0}^{n} |\varphi^{(j)}|^2 \, dx.$$

We observe that the identity (3) is valid for u_ν and w and implies that $Q[u_\nu, w] = 0$, and a fortiori, $Q_J[u_\nu, w] = 0$ for any closed subinterval J of (x_1, x_2), where the subscript J indicates that the integral occurring in the definition of Q is to be evaluated over J only. It is easily verified that $Q_J[u_\nu, w] \to Q_J[u, w]$ as $\nu \to \infty$. Thus we obtain $Q_J[u, w] = 0$. Since J is arbitrary, it follows that $u^{(p-1)}$ is a constant multiple of $w^{(p-1)}$. But this is impossible in the case $n \geqq 2$, since $u^{(p-1)} = 0$ at x_1 and x_2, while $w^{(p-1)}$ is strictly monotone in $[x_1, x_2]$. This completes the proof.

B) On the basis of the identity (7) it is easy to prove a comparison theorem for a class of systems of differential inequalities involving L and M defined by (4) and (5). To simplify the notation, we define the inequality $U > 0\,[U \geqq 0]$ for a real symmetric $N \times N$ matrix U if and only if U is positive definite [U is positive semi-definite].

THEOREM 2 B. *Suppose there exists a nontrivial real-valued vector function* $v \in D_L[x_1, x_2]$ *which satisfies*

$$\int_{x_1}^{x_2} v^T L v \, dx \leqq 0,$$

$$v(x_i) = v'(x_i) = \cdots = v^{(n-1)}(x_i) = 0, \qquad i = 1, 2,$$

$$\int_{x_1}^{x_2} \sum_{j=0}^{n} (v^{(j)})^T (A_j - B_j) v^{(j)} \, dx \geqq 0.$$

If an M-*prepared matrix function* $W \in D_M[x_1, x_2]$ *satisfies*

$$W^T M W \geqq 0 \quad in \quad (x_1, x_2), \qquad where \quad B_n \geqq 0,$$

$$(-1)^k (W^{(k)})^T \sum_{j=k}^{n} (-1)^j (B_j W^{(j)})^{(j-k)} \geqq 0 \quad in \quad (x_1, x_2), \qquad k = 1, \cdots, n-1,$$

$$det \sum_{j=p}^{n} (-1)^j (B_j W^{(j)})^{(j-p)} \neq 0 \quad in \ (x_1, x_2) \quad for\ some \ p \in \{1, \cdots, n-1\},$$

then at least one of $W, W', \cdots, W^{(n-1)}$ *is singular at some point of* (x_1, x_2) *unless* $u = Wc$ *for some constant vector* c *in the case* $n = 1$.

Remark. Theorem 2 B is an extension of a recent result of Kuks [9] for fourth order systems of differential equations.

4. Disconjugacy criteria

A) Consider the scalar ordinary differential equation

$$lu \equiv \sum_{j=0}^{n} (-1)^j (a_j u^{(j)})^{(j)} = 0 \tag{8}$$

in an interval I.

Two points x_1, x_2 of I are called *conjugate with respect to* (8) if there exists a nontrivial solution $u \in D_l[x_1, x_2]$ which satisfies

$$u(x_i) = u'(x_i) = \cdots = u^{(n-1)}(x_i) = 0, \qquad i = 1, 2.$$

The equation (8) is called *disconjugate on* I (or *nonoscillatory on* I) if no two points of I are conjugate with respect to (8). (See for example Coppel [3]).

The following disconjugacy criterion for (8) is an immediate consequence of Theorem 2 A.

THEOREM 3 A. *The equation* (8) *is disconjugate on* I *if there are a differential operator* m *defined by* (2) *and a function* $w \in D_m(I)$ *which satisfy*

$$a_j \geqq b_j \quad in \ I, \quad j = 0, 1, \cdots, n, \quad b_n \geqq 0 \quad in \ I,$$

$$wmw \geqq 0 \quad in \ I, \qquad w \not\equiv 0 \quad in \ I, \tag{9}$$

$$(-1)^k w^{(k)} \sum_{j=k}^{n} (-1)^j (b_j w^{(j)})^{(j-k)} > 0 \qquad in \ I, \qquad k = 1, \cdots, n-1. \tag{10}$$

Example 1 A. As a comparison operator m we take

$$mw \equiv (-1)^n w^{(2n)} - h^{2n} w,$$

where h is a positive constant. It is easy to see that the function $w = \sin h(x - \alpha)$, α being a constant, satisfies (9) and (10) in $(\alpha, \alpha + \pi/2h)$. Theorem 3 A implies therefore that the equation (8) is disconjugate on any interval I of length less than $\pi/2h$ if

$$a_n \geqq 1 \quad , \quad a_j \geqq 0 \quad , \quad j = n-1, \cdots, 1 \quad , \quad a_0 \geqq -h^{2n} \quad \text{in I}.$$

Example 2 A. Let m be the Euler operator

$$mw \equiv (-1)^n (x^{\alpha+n} w^{(n)})^{(n)} - \beta^2 x^{\alpha-n} w,$$

where α and β are constants satisfying

$$\alpha > n - 3 \ (\alpha \neq n - 1) \quad \text{or} \quad \alpha < 1 - n \,, \tag{11}$$

$$\beta^2 < \prod_{j=0}^{n-1} \frac{\alpha^2 - (n - 1 - 2j)^2}{4} \,. \tag{12}$$

A simple computation shows that the function $w = x^\gamma$, $\gamma = (n - 1 - \alpha)/2$, satisfies conditions (9) and (10) in the half-line $(0, \infty)$. It follows from Theorem 3 A that equation (8) is disconjugate on $(0, \infty)$ if

$$a_n \geqq x^{\alpha+n} \quad , \quad a_j \geqq 0 \quad , \quad j = n-1, \cdots, 1 \quad , \quad a_0 \geqq - \beta^2 x^{\alpha-n} \quad \text{in} \quad (0, \infty).$$

B) The disconjugacy of the vector differential equation

$$Lv \equiv \sum_{j=0}^{n} (-1)^j (A_j v^{(j)})^{(j)} = 0 \tag{13}$$

can be defined exactly as in the scalar case. The following disconjugacy criterion follows from Theorem 2 B.

THEOREM 3 B. *The equation* (13) *is disconjugate on* I *if there are a differential operator* M *defined by* (5) *and an* M-*prepared matrix function* $W \in D_M(I)$ *which satisfy*

$$A_j - B_j \geqq 0 \quad in \quad I, \quad j = 0, 1, \cdots, n, \quad B_n \geqq 0 \quad in \quad I,$$

$$W^T M W \geqq 0 \quad in \quad I, \quad det \ W \neq 0 \quad in \quad I,$$

$$(-1)^k (W^{(k)})^T \sum_{j=k}^{n} (-1)^j (B_j W^{(j)})^{(j-k)} > 0 \quad in \quad I, \quad k = 1, \cdots, n-1.$$

Remark. Suppose there are a scalar operator m and a scalar function $w \in D_m(I)$ which satisfy conditions (9) and (10). Assume furthermore that $A_j - b_j E_N \geqq 0$ in I, $j = 0, 1, \cdots, n$. Then the matrix operator M with coefficients $B_j = b_j E_N$, $j = 0, 1, \cdots, n$, and the matrix function $W = wE_N$ clearly satisfy the hypotheses of Theorem 3 B.

Applying Theorem 3 B with Remark and using Examples 1 A and 2 A, we obtain the following disconjugacy criteria for (13).

Example 1 B. Equation (13) is disconjugate on any interval I of length less than $\pi/2h$ if

$$A_n - E_N \geqq 0 \quad , \quad A_j \geqq 0 \quad , \quad j = n-1, \cdots, 1 \quad , \quad A_0 + h^{2n} E_N \geqq 0 \quad in \quad I.$$

Example 2 B. Equation (13) is disconjugate on the half-line $(0, \infty)$ if

$$A_n - x^{\alpha+n} E_N \geqq 0 \quad , \quad A_j \geqq 0 \quad , \quad j = n-1, \cdots, 1 \,,$$

$$A_0 + \beta^2 x^{\alpha-n} E_N \geqq 0 \quad in \quad (0, \infty),$$

where the constants α and β satisfy (11) and (12), respectively.

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