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Weakened field equations in general relativity admitting a generalised Takeno's space-time

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Teorie relativistiche. — *Weakened field equations in general relativity admitting a generalised Takeno's space–time.* Nota di KRISHNA BIHARI LAL e VINAI K. SRIVASTAVA, presentata (*) dal Socio C. CATTANEO.

RIASSUNTO. — Si considerano le equazioni di campo indebolite proposte da vari Autori (Eddington, Buchdahl, Kilmister e Newman, Rund e Du Plessis) in sostituzione delle equazioni gravitazionali nel vuoto di Einstein e se ne studiano soluzioni del tipo generalizzate di Takeno.

1. INTRODUCTION

The vacuum field equation in general relativity is given by

$$R_{ij} = 0 . \tag{1.1}$$

Eddington, Buchdahl, Kilmister and Newman, Rund and Du Plessis have suggested vacuum field equations alternative to (1.1) which are weaker than (1.1) in the sense that they admit a class of solutions for which (1.1) holds, and have called such field equations "weakened field equations". They are given by

$$J_{ikl} \equiv R^{j}_{ikl;j} = 0 \tag{1.2}$$

$$G_{jk} \equiv (-g)^{1/4} \Big[g^{ih} R_{kj;ih} - g^{ih} R_{ij;kh} + \frac{1}{6} R_{;kj} - \\ - \frac{1}{6} g_{jk} g^{ih} R_{;ih} - R^{ih} C_{jhik} + \frac{1}{6} R g^{ij} C_{jhjk} \Big] = 0 \tag{1,3}$$

with the property that

$$G_{ik} = G_{kj}$$

and

$$G^{i}_{k;j} = 0$$

$$E^{hk} \equiv (-g)^{1/2} [g^{hj} g^{ki} \{ 2 R_{jlim} R^{ml} + g^{ml} R_{ij;lm} - R_{;ij} \} - \\ - \tfrac{1}{2} g^{hk} \{ R^{l}_{m} R^{m}_{l} - g^{lm} R_{;lm} \}] = 0 \tag{1.4}$$

with the property that

$$E^{hk} = E^{kh}$$

and

$$E^{hk}_{;h} = 0$$

$$\varepsilon^{rs} \equiv (-g)^{1/2} [(g^{rs} g^{tu} - \tfrac{1}{2} g^{rt} g^{su} - \tfrac{1}{2} g^{ru} g^{st}) R_{;ut} + R (R^{sr} - \tfrac{1}{4} g^{sr} R)] = 0 \tag{1.5}$$

(*) Nella seduta dell'8 marzo 1975.

with the property that

$$\varepsilon^{rs} = \varepsilon^{sr}$$

and

$$\varepsilon^{rs}_{\ \ ;r} = 0$$

and

$$\text{(1.6)} \qquad H^{ij}_{jk} \equiv R^{ij}_{;k} = 0 .$$

D. Lowelock [1] has solved these five weakened field equations for the special line element.

$$\text{(1.7)} \qquad ds^2 = \frac{a^2}{r^2}\,(c^2\,dt^2 - dr^2 - r^2\,d\theta^2 - r^2 \sin^2\theta d\varphi^2)$$

(a being a constant) which for certain reasons he considered to be unphysical.

In this paper we propose to solve the above weakened field equations for the line element

$$\text{(1.8)} \qquad \begin{aligned} ds^2 = {} & -Adx^2 - 2\,Ddx\,dy - Bdy^2 - (C - E)\,dz^2 - \\ & - 2\,Edz\,dt + (C + E)\,dt^2 \end{aligned}$$

where A, B, C, D are functions of the single variable $Z = z - t$ and $E = E(x, y, Z)$.

The metric (1.8) reduces to Takeno's [2] general plane wave metric when $E = E(Z)$, i.e. all the components of the metric tensor are functions of the single variable $Z = z - t$, while if $A = B = C = 1$ $D = 0$ and $E = -2f(x, y, Z)$ it reduces to Peres' [3] metric. The relation between Takeno's metric and Peres' one is made clear in [4].

2. CURVATURE PROPERTIES OF THE METRIC (1.8)

From (1.8) we can easily obtain the non-vanishing components of g^{ij} and the Christoffel symbols $\{{}^{\,i}_{jk}\}$ of the second kind. Using them the non-vanishing components of the curvature tensor R_{ijkl} and the "Ricci-Tensor" R_{ij} are given by

$$\text{(2.1)} \qquad \begin{cases} R_{1313} = -R_{1314} = \quad R_{1414} = L \\ R_{2323} = -R_{2324} = \quad R_{2424} = M \\ R_{1323} = -R_{1324} = -R_{1423} = R_{1424} = N \end{cases}$$

and

$$\text{(2.2)} \qquad \begin{cases} R_{33} = -R_{34} = R_{44} = \dfrac{1}{m}\,(BL - 2\,DN + AM) = \dfrac{\bar{\bar{m}}}{2\,m} - \dfrac{\bar{m}\,\bar{c}}{2\,mc} - \\ \quad - \dfrac{\bar{m}^2}{4\,m^2} - \dfrac{\bar{A}\,\bar{B} - \bar{D}^2}{2\,m} - \dfrac{1}{2\,m}\left(A\,\dfrac{\partial^2 E}{\partial y^2} - 2\,D\,\dfrac{\partial^2 E}{\partial x\,\partial y} + B\,\dfrac{\partial^2 E}{\partial x^2}\right) \end{cases}$$

where

$$(2.3)\quad \begin{cases} 2\,L = \bar{\bar{A}} - \dfrac{\bar{A}\,\bar{C}}{C} - \dfrac{1}{2\,m}(B\bar{A}^2 + A\bar{D}^2 - 2\,\bar{A}D\bar{D}) - \dfrac{\partial^2 E}{\partial x^2} \\ 2\,M = \bar{\bar{B}} - \dfrac{\bar{B}\,\bar{C}}{C} - \dfrac{1}{2\,m}(A\bar{B}^2 + B\bar{D}^2 - 2\,\bar{B}D\bar{D}) - \dfrac{\partial^2 E}{\partial y^2} \\ 2\,N = \bar{\bar{D}} - \dfrac{\bar{D}\,\bar{C}}{C} - \dfrac{1}{2\,m}(A\bar{B}\bar{D} + \bar{A}B\bar{D} - \bar{A}\bar{B}D - D\bar{D}^2) - \dfrac{\partial^2 E}{\partial x\,\partial y} \end{cases}$$

and

$$m = AB - D^2.$$

It is easy to show that the non-vanishing components of R^{ij} and R^i_j will be given by

$$(2.4)\quad \begin{cases} R^{33} = R^{44} = \quad R^{34} = \quad R^{43} = \dfrac{BL - 2\,DN + AM}{m\,C^2} \\ R^3_3 = R^4_3 = -R^3_4 = -R^4_4 = -\dfrac{BL - 2\,DN + AM}{m\,C} \end{cases}$$

and the scalar curvature R is zero.

3. Solutions of the weakened field equations

We shall state the main result of this section in the form of two theorems.

THEOREM 1. *The metric* (1.8) *is a solution of the equations* (1.2), (1.3), (1.4) *and* (1.6) *if*

$$\bar{\bar{m}} - \frac{\bar{m}\,\bar{c}}{C} - \frac{\bar{m}^2}{2\,m} - (\bar{A}\bar{B} - \bar{D}^2) - A\left(\frac{\partial^2 E}{\partial y^2} - 2\,D\,\frac{\partial^2 E}{\partial x\,\partial y} + B\,\frac{\partial^2 E}{\partial x^2}\right) = 2\,m\lambda C^2$$

where λ *is a constant.*

Proof. Using (2.2) the above condition can be put as

$$(3.2)\qquad R_{ij;k} = 0.$$

Solution of Field Equation (1.2).

Using Bianchi's Identity

$$R^j_{ikl;h} + R^j_{ihk;l} + R^j_{ilh;k} = 0$$

equation (1.2) can be put as

$$J_{ikl} \equiv R^j_{ikl;j} = R_{ik;l} - R_{il;k} = 0$$

which is obviously satisfied if (3.2) holds.

Solution of Field Equation (1.3).

We know that the Weyl curvature tensor $\mathbf{C}_{jlim}$ is defined by

$$(3.3) \qquad \mathbf{C}_{jlim} = R_{jlim} - \frac{1}{2}(R_{ji}\, g_{lm} - R_{li}\, g_{jm} - R_{jm}\, R_{li} + R_{lm}\, g_{ij}) + \\ + \frac{R}{6}(g_{ij}\, g_{lm} - g_{li}\, g_{jm}) .$$

Using (2.2), (2.4), (3.2) and (3.3) it is seen that the field equation (1.3) is identically satisfied.

Solution of Field Equation (1.4).

From (2.4) we see that

$$R^l_m \cdot R^m_l = 0 .$$

Using (2.1), (2.2), (2.4), (3.2) and (3.4) we can show that

$$E^{hk} = 0$$

i.e. the field equation (1.4) is satisfied.

Solution of the Field Equation (1.6).

Using (2.2) the equation (1.6) can be put as

$$H^{ij}_k \equiv R^{ij}_{;k} = (g^{i3}\, g^{j3} - g^{i3}\, g^{j4} - g^{i4}\, g^{j3} + g^{i4}\, g^{j4})\, R_{33;k} = 0 .$$

It is obviously satisfied if R_{33} ; K = 0 i.e. the given condition holds.

THEOREM 2. *The metric* (1.8) *is always a solution of equation* (1.5).

Proof. Since for the metric (1.8), the scalar curvature R is zero, equation (1.5) is identically satisfied.

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