Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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A static coordinates and inequalities in Cosserat's continuum

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.3, p. 385–393.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_3_385_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Fisica-matematica. — *Astatic coordinates and inequalities in Cosserat's continuum* (*). Nota di TOMMASO RUGGERI, presentata (**) dal Corrisp. G. GRIOLI.

RIASSUNTO. — Usando una naturale estensione delle coordinate astatiche ed iperastatiche della teoria classica dei continui si stabiliscono delle corrispondenti proprietà di media per lo stato tensionale nel caso dei continui di Cosserat.

La presenza della parte emisimmetrica dello stress restringe le quantità dei valori medi d i prodotti ello stress per c:ordinate che nel caso classico era possibile esprimere in funzione della sollecitazione esterna. Inoltre, non permette di determinare i valori medi delle coppie di contatto.

Tuttavia è possibile stabilire delle limitazioni inferiori per la densità di energia potenziale elastica in funzione della sollecitazione esterna.

La validità del teorema di Clapeyron permette a volte di avere informazioni sulla deformazione stessa del continuo.

Infine, nel caso di rotazioni vincolate, si fà un'applicazione al problema dell'equilibrio di un solido sollecitato da una coppia di braccio nullo e da due coppie concentrate.

1. GENERAL REMARKS

The linearized equations of the statics of the Cosserat continuum are:

$$(1)\quad \begin{cases} \Phi_{ik,k} = \rho_* F_i \\ \Psi_{ik,k} + \varepsilon_{ilm}\Phi_{ml} = \rho_* M_i \end{cases} \text{(in } C_*\text{)};\qquad (2)\quad \begin{cases} \Phi_{ik}\, n_k^* = f_i^* \\ \Psi_{ik}\, n_k^* = m_i^* \end{cases} \text{(in } \sigma_*\text{)}$$

where C_* is the volume of the natural state, σ_* its boundary, Φ_{ik} and Ψ_{ik} are respectively the matrices of the stress and of the couple-stress, **F** and **M** characterize the forces and the moments and $k = \partial/\partial y_k$.

The classical n-hyperastatic coordinates are, as is well known [1]:

$$(3)\qquad b_{\eta\tau\lambda}^{(r)} = -\frac{1}{C_*}\left\{\int_{C_*} \rho_* y_1^\eta y_2^\tau y_3^\lambda F_r\, dC_* + \int_{\sigma_*} y_1^\eta y_2^\tau y_3^\lambda f_r^*\, d\sigma_*\right\}.$$

From (1,1) and (2,1) we have:

$$(4)\qquad b_{\eta\tau\lambda}^{(r)} = \overline{\Phi_{rk}\,(y_1^\eta y_2^\tau y_3^\lambda)_{,k}}$$

where the upper bar denotes the mean value in C* and

$$(5)\qquad \eta + \tau + \lambda = n \qquad (n = 0, 1, 2, \cdots).$$

(*) Work supported by the Gruppo Nazionale per la Fisica Matematica, C.N.R.
(**) Nella seduta dell'8 marzo 1975.

In the same way it is natural to introduce the corresponding n-hyperastatic coordinates related to the moments:

$$\text{(6)} \qquad B^{(r)}_{\gamma\mu\nu} = -\frac{1}{C_*}\left\{\int_{C_*} \rho_* y_1^\gamma y_2^\mu y_3^\nu M_r \, dC_* + \int_{\sigma_*} y_1^\gamma y_2^\mu y_3^\nu m_r^* \, d\sigma_*\right\}$$

with

$$\text{(7)} \qquad \gamma + \mu + \nu = n - 1 \qquad (n \geq 1).$$

Putting:

$$\text{(8)} \qquad C^{(r)}_{\gamma\mu\nu} = \overline{\varepsilon_{rlm} \Phi_{lm} y_1^\gamma y_2^\mu y_3^\nu}$$

we obtain, from (1,2) and (2,2) (taking into account (6) and (8)) the relation (similar to (4)):

$$\text{(9)} \qquad B^{(r)}_{\gamma\mu\nu} - C^{(r)}_{\gamma\mu\nu} = \overline{\Psi_{rk} (y_1^\gamma y_2^\mu y_3^\nu)_{,k}} \, .$$

However we must observe that (for $n = 1$) while eq. (4) determines all the mean values of the Φ_{ik}, eq. (9) does not determine all the mean values of the Ψ_{ik}. In fact in this case eq. (4) does not give all the mean values of the type $\overline{\Phi_{ik} y_r}$ (but only some of them) and this implies the presence of more unknowns in the eq. (9), precisely the $C^{(r)}_{\gamma\mu\nu}$.

Remark I. The cardinal eqs. can be written:

$$\text{(10)} \qquad \begin{cases} b^{(r)}_{000} = 0 \\ B^{(1)}_{000} + b^{(3)}_{010} - b^{(2)}_{001} = 0 \\ B^{(3)}_{000} + b^{(2)}_{100} - b^{(1)}_{010} = 0 \, . \end{cases}$$

2. Some properties of the mean values of the stress

In the following, we consider the cases $n = 1$ and $n = 2$ and therefore we rewrite the n-hyperastatic coordinates in the more convenient form:

$$\text{(11)} \qquad a_{rs} = -\frac{1}{C_*}\left\{\int_{C_*} \rho_* y_s F_r \, dC_* + \int_{\sigma_*} y_s f_r^* \, d\sigma_*\right\}$$

$$\text{(12)} \qquad b_{rst} = -\frac{1}{C_*}\left\{\int_{C_*} \rho_* y_r y_s F_t \, dC_* + \int_{\sigma_*} y_r y_s f_t^* \, d\sigma_*\right\}$$

$$\text{(13)} \qquad B_{rs} = -\frac{1}{C_*}\left\{\int_{C_*} \rho_* y_s M_r \, dC_* + \int_{\sigma_*} y_s m_r^* \, d\sigma_*\right\}$$

$$\text{(14)} \qquad C_{rs} = \overline{\varepsilon_{rlm} \Phi_{lm} y_s}$$

$$\text{(15)} \qquad S_{rst} = \tfrac{1}{2} (b_{str} + b_{rts} - b_{rst}).$$

The relations for the mean values, in this case, are:

(16) $$a_{rs} = \overline{\Phi_{rs}}$$

(17) $$b_{rst} = \overline{y_r\ \Phi_{ts}} + \overline{y_s\ \Phi_{tr}}$$

(18) $$B_{rs} - C_{rs} = \overline{\Psi_{rs}}$$

(19) $$S_{rst} = \overline{y_t\ \Phi_{(rs)}} + \overline{y_s\ \Phi_{[rt]}} + \overline{y_r\ \Phi_{[st]}}\,.$$

Putting $T_{rs} = \Phi_{(rs)}$, from (14) and (19) it follows:

(20) $$\overline{y_t\, T_{rs}} = S_{rst} + A_{rst}$$

with

(21) $$A_{rst} = \tfrac{1}{2}\,(\varepsilon_{mtr}\, C_{ms} + \varepsilon_{mts}\, C_{mr})\,.$$

Remark II. It is easily seen that in (21), it is possible to substitute to C_{ik} its deviator:

$$C^{(D)}_{ik} = C_{ik} - \frac{1}{3}\, C_{jj}\, \delta_{ik}\,, \qquad \text{i.e.,}$$

(22) $$A_{rst} = \tfrac{1}{2}\,(\varepsilon_{mtr}\, C^{(D)}_{ms} + \varepsilon_{mts}\, C^{(D)}_{mr})\,.$$

Furthermore, we observe that $A_{rst} = 0$ when $r = s = t$ and consequently the $\overline{T_{rs}\, y_t}$ with $r = s = t$ are determined in terms of the external forces.

It is not difficult to verify that [2]:

(23) $$\int_{C_*} q_{iklm}\, R_{ik}\, R_{lm}\, dC_* \geq C_*\, q_{iklm} \left(\overline{R_{ik}}\ \overline{R_{lm}} + \sum_{t=1}^{3} \frac{\overline{R_{ik}\, y_t}\ \overline{R_{lm}\, y_t}}{\rho_t^2} \right) \geq$$

$$\geq C_*\, q_{iklm}\, \overline{R_{ik}}\ \overline{R_{lm}}$$

where R_{ik} is a matrix, q_{iklm} are constant the coefficients of a positive definite or semidefinite form and:

(24) $$\rho_t^2 = \frac{1}{C_*} \int_{C_*} y_t^2\, dC_*\,.$$

We suppose that the elastic potential energy of the continuum may be expressed as a sum of two positive forms. The first depending on the T_{ik}, the second on the Ψ_{ik}:

(25) $$W'(T_{rs}) = q'_{iklm}\, T_{ik}\, T_{lm} \quad : \quad W''(\Psi_{rs}) = q''_{iklm}\, \Psi_{ik}\, \Psi_{lm}\,.$$

Using twice (23) with $R_{ik} = T_{ik}$ then with $R_{ik} = \Psi_{ik}$ we get:

(26) $$\int_{C_*} (W' + W'')\, dC_* \geq C_* \left\{ q'_{iklm} \left(\overline{T_{ik}}\ \overline{T_{lm}} + \sum_{t=1}^{3} \frac{\overline{T_{ik}\, y_t}\ \overline{T_{lm}\, y_t}}{\rho_t^2} \right) + q''_{iklm}\, \overline{\Psi_{ik}}\ \overline{\Psi_{lm}} \right\}.$$

The second member depends only on the external forces and on the undetermined parameters C_{mq}.

3. A Lower bound for the elastic potential energy

Now we suppose that the continuum is a Cosserat-continuum with constrained rotations. In this case the potential energy assumes the form [3]:

$$V(\varepsilon_{rs}, \mu_{rs}) = V'(\varepsilon_{rs}) + V''(\mu_{rs}) \tag{27}$$

with

$$V'(\varepsilon_{rs}) = \tfrac{1}{2}\{\lambda(\varepsilon_{jj})^2 + 2\mu\varepsilon_{ik}\varepsilon_{ik}\} \tag{28}$$

$$V''(\mu_{rs}) = \tfrac{1}{2}\left\{\sum_{i,k}{}_{*}(B\mu_{ik} + C\mu_{ki})\mu_{ki} + (B + C)(\mu_{11} + \mu_{22})^2\right\} \tag{29}$$

where the ε_{ik} characterize the deformation-matrix and $\mu_{ik} = \omega_{i,k} = \frac{1}{2}\varepsilon_{iml}s_{l,mk}$; $\boldsymbol{\omega}$ denotes the local rotation and $\boldsymbol{s}$ is the displacement, λ and μ are the Lamé's constants; B and C the structural constants satisfying the inequalities:

$$B > 0\,;\ |C| < B\,. \tag{30}$$

In (29), the term corresponding to $r = s = 3$ is absent in $\sum_{r,s}{}_{*}$ because $\mu_{ii} = 0$ $(\mu_{ik} = \mu_{ik}^{(D)})$.

The constitutive equations are:

$$\begin{aligned} T_{rs} &= -\frac{\partial V'}{\partial \varepsilon_{rs}} = -\lambda\varepsilon_{jj}\,\delta_{rs} - 2\mu\varepsilon_{rs} \\ \Psi_{rs} - \Psi_{33}\,\delta_{rs} &= -\frac{\partial V''}{\partial \mu_{rs}} = -B\mu_{rs} - C\mu_{sr} - (B + C)(\mu_{11} + \mu_{22})\,\delta_{rs}\,. \end{aligned} \tag{31}$$

Classically the (31,1) can be inverted and $V'(\varepsilon_{rs})$ transforms into $W'(T_{rs})$:

$$W'(T_{rs}) = \frac{1}{4\mu}\{T_{ik}T_{ik} + h(T_{ii})^2\} \quad , \quad h = -\frac{\lambda}{3\lambda + 2\mu}\,. \tag{32}$$

By contracting (31,2) with δ_{rs} we have:

$$\Psi_{ii} - 3\Psi_{33} = -3(B + C)(\mu_{11} + \mu_{22})$$

substituting in (31,2) we find:

$$\Psi_{rs}^{(D)} = -B\mu_{rs} - C\mu_{sr} \qquad \text{therefore} \quad \mu_{rs} = \frac{1}{B^2 - C^2}(C\Psi_{sr}^{(D)} - B\Psi_{rs}^{(D)}).$$

Inserting this value in (29), rewritten in the form:

$$V'' = \tfrac{1}{2}(B\mu_{rs}^{(D)} + C\mu_{rs}^{(D)})\,\mu_{sr}^{(D)},$$

we finally obtain:

$$W''(\Psi_{rs}) = \frac{1}{2(B^2 - C^2)} (B\Psi_{rs}^{(D)} - C\Psi_{sr}^{(D)}) \Psi_{rs}^{(D)}. \tag{33}$$

From (25), (32) and (33) it follows:

$$q'_{iklm} = \frac{1}{4\mu} (\delta_{il} \delta_{km} + h\delta_{ik} \delta_{lm}) \tag{34}$$

$$q''_{iklm} = \frac{1}{2(B^2 - C^2)} (B\delta_{il} \delta_{km} - C\delta_{im} \delta_{kl}) \tag{35}$$

(26) is analogous to the relation obtained in the classical case with symmetric stress, but while in this case the second member is completely determined, it now depends on the unknowns C_{rs}.

Proceeding as in the classical theory in presence of unknowns constraints, it is natural to choose C_{rs} in such a way that the second member in (26) is minimum. We note that the minimum cannot be zero (see Remark II).

Putting

$$Q_{ikt} = S_{ikt} + A_{ikt} \tag{36}$$

$$G_{iklm} = \overline{T_{ik}}\, \overline{T_{lm}} + \sum_{t=1}^{3} \frac{1}{\rho_t^2} Q_{ikt} Q_{lmt} \tag{37}$$

$$H(C_{pq}^{(D)}) = q'_{iklm} G_{iklm} \tag{38}$$

$$L(C_{pq}^{(D)}) = q''_{iklm} (B_{ik}^{(D)} - C_{ik}^{(D)}) (B_{lm}^{(D)} - C_{lm}^{(D)}) \tag{39}$$

the minimum condition is:

$$\frac{\partial H}{\partial C_{pq}^{(D)}} + \frac{\partial L}{\partial C_{pq}^{(D)}} = 0$$

We have:

$$\frac{\partial H}{\partial C_{mq}^{(D)}} = q'_{iklj} \frac{\partial G_{iklj}}{\partial A_{rst}} \frac{\partial A_{rst}}{\partial C_{mq}^{(D)}} = \frac{1}{2\mu} \sum_{t=1}^{3} \frac{1}{\rho_t^2} \{h\, Q_{llt}\, \varepsilon_{mtq} + Q_{iqt}\, \varepsilon_{mti}\}$$

$$\frac{\partial L}{\partial C_{mq}^{(D)}} = -\frac{1}{B^2 - C^2} \{B(B_{mq}^{(D)} - C_{mq}^{(D)}) - C(B_{qm}^{(D)} - C_{qm}^{(D)})\}$$

thus

$$\frac{1}{2\mu} \sum_{t=1}^{3} \frac{1}{\rho_t^2} \{hQ_{llt}\, \varepsilon_{mtq} + Q_{iqt}\, \varepsilon_{mti}\} - \tag{40}$$

$$- \frac{1}{B^2 - C^2} \{B(B_{mq}^{(D)} - C_{mq}^{(D)}) - C(B_{qm}^{(D)} - C_{qm}^{(D)})\} = 0.$$

By contracting with ε_{qmr} we find:

$$\frac{1}{2\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}\{(1+2h)\,Q_{llt}\,\delta_{rt}-Q_{rtt}\}-\frac{1}{B-C}\,\varepsilon_{qmr}\,(B_{mq}^{(D)}-C_{mq}^{(D)})=0, \tag{41}$$

while the symmetric part of (40) gives:

$$\frac{1}{2\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}\{Q_{iqt}\,\varepsilon_{mti}+Q_{imt}\,\varepsilon_{qti}\}-\frac{2}{B+C}\,(B_{(mq)}^{(D)}-C_{(mq)}^{(D)})=0. \tag{42}$$

Putting now:

$$S_r=\frac{1}{2\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}\{S_{rtt}-(1+2h)\,S_{llt}\,\delta_{rt}\}+\frac{1}{B-C}\,\varepsilon_{qmr}\,B_{mq}^{(D)} \tag{43}$$

$$s_{mq}=-\frac{1}{2\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}(S_{iqt}\,\varepsilon_{mti}+S_{imt}\,\varepsilon_{qti})+\frac{2}{B+C}\,B_{(mq)}^{(D)} \tag{44}$$

$$(s_{qq}=0\,;\;s_{mq}=s_{(mq)}=s_{mq}^{(D)})\,.$$

(41) and (42) become:

$$\frac{1}{2\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}\{(1+2h)\,A_{llt}\,\delta_{rt}-A_{rtt}\}+\frac{1}{B-C}\,\varepsilon_{qmr}\,C_{mq}^{(D)}=S_r \tag{45}$$

$$\frac{1}{2\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}\{A_{iqt}\,\varepsilon_{mti}+A_{imt}\,\varepsilon_{qti}\}+\frac{2}{B+C}\,C_{(mq)}^{(D)}=s_{mq}\,. \tag{46}$$

Remembering the definition (22) we get:

$$\frac{1}{2\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}\{(1+2h)\,\varepsilon_{mtq}\,C_{mq}^{(D)}\,\delta_{rt}-\tfrac{1}{2}\,\varepsilon_{mtr}\,C_{mt}^{(D)}\}+\frac{1}{B-C}\,\varepsilon_{mrq}\,C_{mq}^{(D)}=S_r \tag{47}$$

$$\frac{1}{4\mu}\sum_{t=1}^{3}\frac{1}{\rho_t^2}\{2\,C_{(mq)}^{(D)}\,\delta_{tt}-C_{tq}^{(D)}\,\delta_{mt}-C_{tm}^{(D)}\,\delta_{tq}+C_{ji}^{(D)}\,(\varepsilon_{mti}\,\varepsilon_{jtq}+\varepsilon_{qti}\,\varepsilon_{jtm})+ \tag{48}$$

$$+\frac{2}{B+C}\,C_{(mq)}^{(D)}=s_{mq}\,.$$

These eqs. determine the components of C_{mq} in terms of S_r and s_{mq}.
For the sake of simplicity we give the solution when in C_* the central ellipsoid of inertia is spherical ($\rho_t^2=\rho^2\;\forall t=1,2,3$).
In this case (47) and (48) reduce to:

$$\frac{1}{2k}\,\varepsilon_{mrq}\,C_{mq}^{(D)}=S_r \tag{49}$$

$$\frac{1}{b}\,C_{(mq)}^{(D)}=s_{mq} \tag{50}$$

where

(51)
$$\frac{1}{2k} = \frac{3+4h}{4\mu\rho^2} + \frac{1}{B-C}$$

(52)
$$\frac{1}{2b} = \frac{3}{4\mu\rho^2} + \frac{1}{B+C}.$$

Thus the minimizing expressions of the second member of (26) are:

(53)
$$C_{mq}^{(D)} = bs_{mq} + k\varepsilon_{rqm} S_r .$$

Consequently we have:

(54)
$$A_{rst} = -\frac{b}{4\mu\rho^2}\{2 S_{rst} - S_{tsr} - S_{trs} + S_{imj}(\varepsilon_{sji}\varepsilon_{mtr} + \varepsilon_{mts}\varepsilon_{rji})\} + \\ + \frac{k}{2}(2 S_t \delta_{rs} - S_r \delta_{ts} - S_s \delta_{tr}) + \frac{b}{B+C}(\varepsilon_{mtr} B_{ms}^{(D)} + \varepsilon_{mts} B_{mr}^{(D)})$$

with

(55)
$$S_r = \frac{1}{2\mu\rho^2}(S_{rll} - (1+2h) S_{llr}) + \frac{1}{B-C}\varepsilon_{qmr} B_{mq}^{(D)}.$$

4. AN APPLICATION OF THE PRECEDING THEORY

We consider an elastic homogeneous and isotropic solid sollicitated by two forces in equilibrium. Let l be the distance between the points K′ and K″ of applications of the forces, Q the middle point, Δl the value of the lengthening of the fibre K′ K″, f the intensity of the couple with the sign + or — corresponding to traction or pressure.

We take the reference frame G, y_1, y_2, y_3 where G is the center of mass of C_* and its axes are the principal axes of inertia. Let $K' \equiv (y_r')$, $K'' \equiv (y_r'')$, $Q \equiv (x_r)$ and $\boldsymbol{u} = \dfrac{K'K''}{|K'K''|} \equiv (\varphi_i)$; $\boldsymbol{x} = GQ$, $x^2 = \boldsymbol{x}\cdot\boldsymbol{x}$ and $\alpha = \boldsymbol{x}\cdot\boldsymbol{u}$.

We easily obtain (see [4]):

(56)
$$-C_* a_{rs} = fl\,\varphi_r\,\varphi_s$$

(57)
$$-C_* S_{rst} = fl\,\varphi_r\,\varphi_s\,x_t$$

(58)
$$B_{rs} = 0 .$$

Taking into account (53) and (54), eq. (26) after some calculations becomes:

(59)
$$\int_{C_*} W\,dC_* \geq \frac{f^2 l^2}{C_*}\left\{\frac{1}{2E}\left(1 + \frac{x^2}{\rho^2}\right) + \frac{1}{2E\rho^2}[(p^2+r)\alpha^2 - (p^2+q^2)x^2]\right\}$$

with

$$p^2 = \frac{(B+C)(1+\sigma)^2}{3(1+\sigma)(B+C)+2\rho^2 E} \geq 0 \tag{60}$$

$$q^2 = \frac{(B-C)(1-\sigma)^2}{(3-\sigma)(B-C)+2\rho^2 E} > 0 \tag{61}$$

$$r = \frac{(1-3\sigma)(1+\sigma)(B-C)}{(3-\sigma)(B-C)+2\rho^2 E} \tag{62}$$

where E and σ are respectively the Young modulus and the Poisson coefficient.

The first term of the second member of (59) coincides exactly with the classical one.

However we observe that (59) cannot give the classical results if $B = C = 0$, because the theory in this case is no more valid. In fact if $B = C = 0$ the quadratic form which expresses the density of elastic potential energy, considered as a function of all the variables ε_{rs} and μ_{rs} is not positive definite and the relations stress-strain cannot be inverted (see (30)).

Remembering also that $-1 \leq \sigma \leq 1/2$, $E > 0$ and that $r < q^2$, $\alpha^2 \leq x^2$, we can see that the inequality (59) has the second member smaller than the classical one.

This could be expected, for in this case we substantially have a larger structural deformability of the continuum and consequently the presence of the unknown parameters C_{rs}.

A still greater difference with the classical theory appears when besides the two forces also two concentred couples exist in the points K′ and K″ with the respective moments mk and $-mk$ ($m \geq 0$, $k^2 = 1$, $k \cdot u = 0$).

In this case the B_{rs} are not zero; in fact:

$$-C_* B_{rs} = ml\, k_r\, k_s\,. \tag{63}$$

The inequality for the energy now becomes:

$$\int_{C_*} W\, dC_* \geq \frac{l^2}{C_*^2} \left\{ f^2 \left[\frac{1}{2E} \left(1 + \frac{x^2}{2} \right) + \frac{1}{2E\rho^2} [(p^2+r)\alpha^2 - (p^2+q^2)x^2] \right] + \right. \tag{64}$$
$$\left. + 4m^2 \frac{\rho^4 E^2}{[(B+C)(1+\sigma)+2\rho^2 E]^2} \right\}.$$

It is clear that the second member of (64) may be larger than the corresponding classical term.

Remark III. In the theory of Cosserat's continuum with contrained rotation the Clapeyron theorem holds:

$$2 \int_{C_*} W\, dC_* = \delta L \tag{65}$$

where δL is the work done by the external forces in the displacement from the unstressed state to the actual state of equilibrium.

By using (59) and (64) eq. (65) can give informations on the strain of the body.

In the just considered case we have:

$$\delta L = f \mid \Delta l \mid + m \mid \Delta\Omega \mid \tag{66}$$

where $\mid \Delta\Omega \mid$ denotes the modulus of the relative rotation.

If $m = 0$ eqs. (64)-(66) yield a lower bound for $\mid \Delta l \mid$. On the contrary if $f = 0$ we have a lower bound for $\mid \Delta\Omega \mid$.

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