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## On some fixed point theorems with applications to the nonarchimedean Menger spaces

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**Topologia.** — *On some fixed point theorems with applications to the nonarchimedean Menger spaces.* Nota di IOANA ISTRĂŢESCU, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questo lavoro si danno alcuni teoremi sui punti fissi per gli spazi uniformi non archimedi. Utilizzando questi risultati si ottengono teoremi di punto fisso per le applicazioni localmente contrattive per gli spazi di Menger non archimedei.

1. Probabilistic metric spaces were first introduced by K. Menger in 1942 [6] and further developed by him in the early 1950's [7], [8]. Important contributions were made in this field by B. Schweizer and A. Sklar [11].

In [12], [13] some generalizations are given of some classical fixed point theorems for contraction mappings defined on complete probabilistic metric spaces. In [1] it is shown that a collection of pseudometrics $\{d_\alpha\}$ can be defined which generates the usual structure for Menger spaces and fixed point theorems are obtained for Menger spaces by first proving analogous theorems in the more usual setting of a uniform space generated by a collection of pseudometrics.

The aim of this note is to give an extension of some results of G. L. Cain and R. H. Kasriel [1] for nonarchimedean uniform spaces and their applications to nonarchimedean Menger spaces.

2. Let S be an abstract space and $\mathcal{D}$ be a family of nonarchimedean pseudometrics, i.e., $\rho_\alpha \in \mathcal{D}$ iff

1) $\rho_\alpha(x, x) \geq 0$ for every $x \in S$,

2) $\rho_\alpha(x, y) = \rho_\alpha(y, x)$ for all $x, y \in S$,

3) $\rho_\alpha(x, y) \leq \max\{\rho_\alpha(x, y, \rho_\alpha(y, z)\}$ for all $x, y, z \in S$.

The nonarchimedean uniformity generated by $\mathcal{D}$ is obtained by taking as a subbase all sets in $S \times S$ of the form $U_{\alpha,\varepsilon} = \{(x, y) : \rho_\alpha(x, y) < \varepsilon\}$ where $\rho_\alpha \in \mathcal{D}$ and $\varepsilon > 0$. We suppose that for $x \neq y$ in S there is at least one $\rho_\alpha \in \mathcal{D}$ for which $\rho_\alpha(x, y) > 0$, so that the uniformity generated by $\mathcal{D}$ is Hausdorff.

The next theorem is a similar result as in [1] and [9] for the nonarchimedean case.

THEOREM 2.1. *Let* S *be sequentially complete and* $f$ *a mapping of* S *into* S *having the property that for every* $\rho_\alpha \in \mathcal{D}$, *there is a constant* $k_\alpha \in (0, 1)$ *such that* $\rho_\alpha(f(x), f(y)) \leq k_\alpha \rho_\alpha(x, y)$ *for all* $x, y \in S$. *Then there is a unique point* $z \in S$ *such that* $f(z) = z$ *and* $f^n(x) \to z$ *for any* $x \in S$.

(*) Nella seduta dell'8 marzo 1975.

*Proof.* To prove the existence of the fixed point, consider an arbitrary $x \in S$ and define $x_n = f^n(x)$, $n = 1, 2, \cdots$. We show that the sequence $\{x_n\}$ is fundamental in S.

Let U be an arbitrary member of the basis, that is $U = \bigcap_{i=1}^{m} U_{\alpha_i}, \varepsilon_i$. For each $i$, $1 \leq i \leq m$ we have

$$\rho_{\alpha_i}(x_n, x_{n+p}) \leq \max\{\max\{\cdots \max\{\{\max\{\rho_{\alpha_i}(x_n, x_{n+1}), \rho_{\alpha_i}(x_{n+1}, x_{n+2})\},$$
$$\cdots \rho_{\alpha_i}(x_{n+p-2}, x_{n+p-1})\}, \rho_{\alpha_i}(x_{n+p-1}, x_{n+p})\} \leq k^n_{\alpha_i} \rho_{\alpha_i}(x, f(x)).$$

Therefore, if we choose $n$ sufficiently large, we have

$$\rho_{\alpha_i}(x_n, x_{n+p}) < \varepsilon_i \quad \text{for} \quad 1 \leq i \leq m$$

and

$$(x_n, x_{n+p}) \in U.$$

Hence $\{x_n\}$ is a fundamental sequence and since S is sequentially complete, there is a limit $z \in S$, i.e., $f^n(x) \to z$. On the other hand we have

$$\rho_\alpha(f^n(z), f(z)) \leq k_\alpha(f^{n-1}(z), z)$$

for each $\rho_\alpha \in \mathcal{D}$. Thus $f^n(z) \to z$ and since S is Hausdorff $f(z) = z$.

To prove the uniqueness of the fixed point we observe that if there exists $x \neq y$ such that $f(x) = x$ and $f(y) = y$ then

$$0 \leq \rho_\alpha(x, y) = \rho_\alpha(f(x), f(y)) \leq k_\alpha \rho_\alpha(x, y)$$

and therefore $\rho_\alpha(x, y) = 0$ since $k_\alpha \in (0, 1)$. From the fact that S is Hausdorff it follows that $x = y$.

Our next theorem is a nonarchimedean uniform version of a metric result of Edelstein [2].

DEFINITION 2.1. *S is $(\rho_\lambda, \varepsilon)$ chainable if for $\rho_\lambda \in \mathcal{D}$ and $\varepsilon > 0$, and any points $x, y \in S$, there is a finite sequence $x = x_0, x_1, x_2, \cdots, x_n = y$ of elements of S such that $\rho_\lambda(x_{i-1}, x_i) < \varepsilon$ for all $i$, $1 \leq i \leq n$.*

THEOREM 2.2. *Let S be sequentially complete and $f$ a mapping of S into S. Suppose there is a $\rho_\lambda \in \mathcal{D}$ and $\varepsilon > 0$ such that S is $(\rho_\lambda, \varepsilon)$ chainable. If for each $\rho_\alpha \in \mathcal{D}$, there is a constant $k_\alpha \in (0, 1)$ such that $\rho_\alpha(f(x), f(y)) \leq \leq k_\alpha \rho_\alpha(x, y)$ whenever $\rho_\lambda(x, y) < \varepsilon$, then there is a unique $z \in S$ for which $f(z) = z$ and $f^n(x) \to z$ for any $x \in S$.*

The proof is similar as in [1].

THEOREM 2.3. *Let S be sequentially compact and $f$ a mapping of S into S. If there is a $\rho_\lambda \in \mathcal{D}$ and an $\varepsilon > 0$ so that $\rho_\alpha(f(x), f(y)) \leq \rho_\alpha(x, y)$ for all $\rho_\alpha \in \mathcal{D}$ whenever $\rho_\lambda(x, y) < \varepsilon$, then $\rho_\alpha(f(x), f(y)) = \rho_\alpha(x, y)$ whenever $\rho_\lambda(x, y) < \varepsilon$.*

*Proof.* As in [1] we consider $a_1, b_1 \in S$ and such that $\rho_\lambda(a_1, b_1) < \varepsilon$. Construct sequences $\{a_i\}$ and $\{b_i\}$ so that $f(a_i) = a_{i-1}$ and $f(b_i) = b_{i-1}$,

$i = 1, 2, \cdots$. Using the existence of convergent subsequences $\{a_{n_i}\}$ and $\{b_{n_i}\}$ of $\{a_i\}$ and $\{b_i\}$ respectively, we may choose $n_i$ as in [1] such that

$$\rho_\lambda(a_0, a_k) < \delta < \varepsilon \quad , \quad \rho_\lambda(b_0, b_k) < \delta < \varepsilon$$
$$\rho_\alpha(a_0, a_k) < \delta < \varepsilon \quad , \quad \rho_\alpha(b_0, b_k) < \delta < \varepsilon$$

where $k = n_{i+1} - n_i$ and $\delta < \min\{\varepsilon, \rho_\lambda(a_0, b_0)\}$. Then we have

$$\rho_\lambda(a_k, b_k) \le \max\{\rho_\lambda(a_0, a_k), \max\{\rho_\lambda(a_0, b_0), \rho_\lambda(b_0, b_k)\}\} < \rho_\lambda(a_0, b_0) < \varepsilon$$

and therefore

$$\rho_\alpha(a_{k-1}, b_{k-1}) = \rho_\alpha(f(a_k), f(b_k)) \le \rho_\alpha(a_k, b_k)$$

and

$$\rho_\lambda(a_{k-1}, b_{k-1}) \le \rho_\lambda(a_k, b_k) < \varepsilon .$$

Continuing, we have

$$\rho_\alpha(a_1, b_1) \le \rho_\alpha(a_k, b_k) .$$

Also

$$\rho_\alpha(a_1, b_1) \le \max\{\rho_\alpha(a_k, a_0), \max\{\rho_\alpha(a_0, b_0), \rho_\alpha(b_0, b_k)\}\} \le$$
$$\le \max\{\rho_\alpha(a_0, b_0), \delta\} \le \rho_\alpha(a_0, b_0)$$

since $\delta$ is arbitrarily small. Thus

$$\rho_\alpha(a_0, b_0) = \rho_\alpha(f(a_1), f(b_1)) \le \rho_\alpha(a_1, b_1) \le \rho_\alpha(a_0, b_0)$$

and the theorem is proved.

*Remark 1.* Using the same arguments as in [1] and the above theorem we can prove that if $f: S \rightarrow f(S) \subset S$ is a mapping such that there is a $\rho_\lambda \in \mathcal{D}$ and an $\varepsilon > 0$ such that $\rho_\alpha(f(x), f(y)) < \rho_\alpha(x, y)$ for all $\rho_\alpha \in \mathcal{D}$ whenever $\rho_\lambda(x, y) < \varepsilon$ and $\rho_\alpha(x, y) > 0$, then the set $P = \{x \in S, \exists p \in N, f^p(x) = x\}$ is finite and nonempty and for every $y \in S$, $\lim_{k\to\infty} f^{pk}(y) \rightarrow x$ for some $x \in P$.

If S is $(\rho_\lambda, \varepsilon)$ chainable then P consists of exactly one point $x$, the unique fixed point of $f$, and $f^n(y) \rightarrow x$ for every $y \in S$.

3. In this section we give some similar results for the nonarchimedean Menger space.

The terminology and notations for nonarchimedean Menger spaces is as in [11] and [3].

It is known [3] that if $(S, \mathcal{F}, T)$ is a nonarchimedean Menger space with $T = \text{Min}$ the collection of subsets of $S \times S$ defined by $U(\varepsilon, \lambda) = \{(x, y): F_{xy}(\varepsilon) > 1 - \lambda, \varepsilon > 0, \lambda > 0\}$ is a basis for a Hausdorff nonarchimedean uniformity on S.

Let $\rho_\alpha : S \times S \to R$, $\alpha \in (0, 1)$ be defined by

$$\rho_\alpha(x, y) = \sup\{t : F_{xy}(t) \leq 1 - \alpha\}.$$

A similar result as in [1] for the nonarchimedean case is

PROPOSITION 3.1. *The function* $\rho_\alpha$ *is a nonarchimedean pseudometric for each* $\alpha \in (0, 1)$.

*Proof.* Since $F_{xy}$ is nondecreasing, left continuous, $F_{xy}(0) = 0$, $\sup F_{xy}(t) = 1$, $F_{xy} = F_{yx}$ it follows that $\rho_\alpha(x, y)$ is nonnegative, finite and symmetric. Also we have $\rho_\alpha(x, y) \leq \max\{\rho_\alpha(x, z), \rho_\alpha(z, y)\}$ since if there exists $x, y, z \in S$ such that $\rho_\alpha(x, y) > \max\{\rho_\alpha(x, z), \rho_\alpha(z, y)\}$, we can choose $t_1 > \rho_\alpha(x, z)$ and $t_2 > \rho_\alpha(z, y)$ such that $\max\{t_1, t_2\} < \rho_\alpha(x, y)$. Hence

$$F_{xy}(t_1) = 1 - \alpha_1 > 1 - \alpha \qquad \text{and} \quad F_{zy}(t_2) = 1 - \alpha_2 > 1 - \alpha.$$

Then we have

$$1 - \alpha \geq F_{xy}(\max\{t_1, t_2\}) \geq T(F_{xz}(t_1), F_{zy}(t_2)) > T(1 - \alpha, 1 - \alpha) = 1 - \alpha$$

which is a contradiction and the proposition is proved.

*Remark 2.* We have that $F_{xy}(\varepsilon) > 1 - \alpha$ if and only if $\rho_\alpha(x, y) < \varepsilon$ and the topology induced by the family $\{\rho_\alpha\}$ is the same as the $(\varepsilon, \lambda)$ topology [11].

Also since for fixed $x$ and $y$, $\rho_\alpha(x, y)$ is a nonincreasing continuous function of $\alpha$ then for any $\delta > 0$, the family $\{\rho_\alpha, \alpha \in (0, \delta)\}$ generates the same topology generated by the entire family $\{\rho_\alpha\}$ [1].

As in [1] we can prove that if S is a Hausdorff space with a topology generated by a family of nonarchimedean pseudometrics $\{\rho_\alpha : \alpha \in (0, 1)\}$ such that for $x, y \in S$ and $\rho_\alpha(x, y)$ is a nonincreasing continuous function of $\alpha$ then for

$$\mathscr{F}(x, y) = F_{xy}(t) = \int_0^1 H(t - \rho_\alpha(x, y))\, d\alpha$$

(where $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t \leq 0$) we have that $(S, \mathscr{F}, \text{Min})$ is a nonarchimedean Menger space. The family $\{\rho_\alpha\}$ is the same family of pseudometrics associated to the nonarchimedean probabilistic metric $\mathscr{F}$. In what follows we consider that $(S, \mathscr{F}, T)$ is a nonarchimedean Menger space with $T = \text{Min}$.

THEOREM 3.1. *Let* $(S, \mathscr{F}, T)$ *be a complete space,* $f : S \to S$ *and suppose that there is a* $\delta \in (0, 1)$ *so that for each* $\alpha \in (0, \delta)$ *there is a* $k_\alpha \in (0, 1)$ *so that for* $x, y \in S$

$$F_{f(x)f(y)}(k_\alpha t) \geq F_{xy}(t)$$

*whenever* $F_{xy}(t) > 1 - \alpha$. *Then* $f$ *has a unique fixed point* $z$ *and* $f^n(x) \to z$ *for every* $x \in S$.

*Proof.* Let $\{\rho_\alpha\}$ be the family of nonarchimedean pseudometrics associated with $\mathscr{F}$. If $\alpha \in (0, \delta)$, $x, y \in S$ and $\varepsilon > 0$ we have that

$$F_{xy}(t) > 1 - \alpha$$

for every $t = \rho_\alpha(x, y) + \varepsilon$. Then from the Remark 2 we obtain that

$$\rho_\alpha(f(x), f(y)) < k_\alpha t \leq k_\alpha \rho_\alpha(x, y)$$

and the theorem follows from Theorem 2.1.

COROLLARY (Theorem 2.2 [4]). *If there is a constant* $k \in (0, 1)$ *such that* $F_{f(x)f(y)}(kt) \geq F_{xy}(t)$ *for all* $t > 0$ *then* $f$ *has a unique fixed point* $z$ *and* $f^n(x) \to z$ *for every* $x \in S$.

DEFINITION 3.1 [12]. *The nonarchimedean probabilistic metric space* $(S, \mathscr{F})$ *is said to be* $(\varepsilon, \lambda)$ *chainable if for each* $x, y \in S$ *there is a finite set* $x = x_0, x_1, \cdots, x_n = y$ *of* $S$ *such that* $F_{x_{i-1}x_i}(\varepsilon) > 1 - \lambda$ *for* $i = 1, 2, \cdots, n$.

THEOREM 3.2. *Let* $(S, \mathscr{F}, T)$ *be a complete* $(\varepsilon, \lambda)$ *chainable space for some* $\lambda$ *and* $\varepsilon$. *If there exists a* $\delta \in (0, 1)$ *so that for each* $\alpha \in (0, \delta)$, *there is a constant* $k_\alpha \in (0, 1)$ *such that* $F_{xy}(\varepsilon) > 1 - \lambda$ *implies*

$$F_{f(x)f(y)}(k_\alpha t) \geq F_{xy}(t)$$

*whenever* $F_{xy}(t) > 1 - \alpha$, *then* $f$ *has a unique fixed point* $z$ *and* $f^n(x) \to z$ *for all* $x \in S$.

For the proof we use Theorem 2.2 and we obtain as a corollary Theorem 2.3 [4].

Also using the results given in Remark 1 we can prove Theorem 3.5 and Theorem 3.6 of Cain and Kasriel [1] for the nonarchimedean Menger spaces.

## REFERENCES


[1] G. L. CAIN JR. and R. H. KASRIEL (1973) – *Fixed and periodic points of local contraction mappings on probabilistic metric spaces* (to appear in «Math. Systems Theory, Notices Amer. Math. Soc.», *20*, A-675, 709-G 22).

[2] M. EDELSTEIN (1961) – *An extension of Banach's contractive principle*, «Proc. Amer. Math. Soc.», *12*, 7–10.

[3] I. ISTRĂȚESCU and N. CRIVĂȚ (1974) – *On some classes of nonarchimedean probabilistic metric spaces*, Seminar de Spații Metrice Probabiliste, Univ. Timișoara, N. 12.

[4] I. ISTRĂȚESCU (1974) – *Fixed point theorems for some classes of contraction mappings on nonarchimedean probabilistic metric spaces*, Seminar de Spații Metrice Probabiliste, Univ. Timișoara, N. 21.

[5] N. ISTRĂȚESCU (1974) – *Introducere în teoria spațiilor metrice probabiliste cu aplicații*, Ed. Tehnică, București.

[6] K. MENGER (1942) – *Statistical metrics*, « Proc. Nat. Acad. Sci. U.S.A. », *28*, 535–537.

[7] K. MENGER (1951) – *Probabilistic Geometry*, « Proc. Nat. Acad. Sci. U.S.A. », *37*, 226–229.

[8] K. MENGER (1954) – *Géométrie Générale* (Chap. VII), Mémorial des Sciences Mathématiques, N. 124, Paris.

[9] A. F. MONNA (1961) – *Sur un théoréme de M. Luxemburg concernant les points fixed d'une classe d'applications d'un espace metrique dans lui même*, « Indagationes Mathematicae », *64*, 89–96.

[10] C. ONICESCU (1964) – *Nombres et Systèmes aléatoires*, Edition de l'Acad. R.P.R., Bucarest et Ed. Eyrolles, Paris.

[11] B. SCHWAIZER and A. SKLAR (1960) – *Statistical metric spaces*, « Pacific J. Math. », *10*, 313–334.

[12] W. M. SEHGAL and A. T. BHARUCHA–REID (1972) – *Fixed points of contraction mappings on probabilistic metric spaces*, «Math. Systems Theory», 6, 97–102.

[13] H. SHERWOOD (1971) – *Complete probabilistic metric spaces*, «Z. Wahrscheinlichkeitstheorie verw Geb. », *20*, 117–128.