Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Takashi Noiri

Sequentially subcontinuous functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.3, p. 370–373.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_3_370_0>

Topologia. — *Sequentially subcontinuous functions.* Nota di

TAKASHI NOIRI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si introducono le funzioni dette successionalmente sottocontinue e si studiano alcune relazioni fra esse e le funzioni successionalmente continue. Si dimostra che una funzione successionalmente sottocontinua risulta successionalmente continua se il suo grafico è successionalmente chiuso.

1. INTRODUCTION

In [2], by using nets R. V. Fuller introduced the concept of subcontinuity as a generalization of continuity and investigated several relations between a subcontinuous function and functions with the following properties: (1) preserving compact sets; (2) having the closed graph. In the present paper, by using sequences we shall define a new class of functions said to be sequentially subcontinuous and obtain some properties analogous to that of subcontinuous functions.

Throughout this paper, X and Y represent topological spaces and $f: X \to Y$ denotes a function (not necessarily continuous) f of a space X into a space Y. By $x_n \to x$ we denote a sequence $\{x_n\}$ converging to a point x. Let A be a subset of a space X, $\{x_n\}$ a sequence in A and x a point in A. Then it is obvious that $\{x_n\}$ converges to x with respect to X if and only if $\{x_n\}$ converges to x with respect to the subspace A. Therefore, henceforward we shall use "$x_n \to x$" without indicating the distinction.

2. DEFINITIONS

(1) A function $f: X \to Y$ is said to be *sequentially nearly-continuous* if for each point $x \in X$ and each sequence $\{x_n\}$ in X converging to x, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(x_{n_k}) \to f(x)$.

(2) A function $f: X \to Y$ is said to be *sequentially subcontinuous* if for each point $x \in X$ and each sequence $\{x_n\}$ in X converging to x, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $y \in Y$ such that $f(x_{n_k}) \to y$.

(3) A subset A of a space X is said to be *sequentially compact* if every sequence in A has a subsequence converging to a point in A, and *sequentially closed* if no sequence in A converges to a point in X — A.

(4) A function $f: X \to Y$ is said to be *sequentially compact preserving* if the image $f(K)$ of every sequentially compact set K of X is sequentially compact in Y.

(*) Nella seduta dell'8 marzo 1975.

(5) A space X is said to be *semi-Hausdorff* [4] if every sequence in X has at most one limit.

Remark 1. (1) If Y is a sequentially compact space, then every function $f: X \to Y$ is sequentially subcontinuous.

(2) Let $f: X \to Y$ be a function. If for each point $x \in X$ there exists a neighborhood V of x such that $f(V)$ is sequentially compact in Y, then f is sequentially subcontinuous.

3. SEQUENTIALLY SUBCONTINUOUS FUNCTIONS

THEOREM 1. *Every sequentially nearly-continuous function is sequentially compact preserving.*

Proof. Suppose $f: X \to Y$ is a sequentially nearly-continuous function and let K be any sequentially compact set of X. We shall show that $f(K)$ is a sequentially compact set of Y. Let $\{y_n\}$ be any sequence in $f(K)$. Then, for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $\{x_n\}$ is a sequence in the sequentially compact set K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x \in K$. By hypothesis, f is sequentially nearly-continuous and hence there exists a subsequence $\{x_j\}$ of $\{x_{n_k}\}$ such that $f(x_j) \to f(x)$. Thus, there exists a subsequence of $\{y_n\}$ converging to $f(x) \in f(K)$. This shows that $f(K)$ is sequentially compact in Y.

THEOREM 2. *Every sequentially compact preserving function is sequentially subcontinuous.*

Proof. Suppose $f: X \to Y$ is a sequentially compact preserving function. Let x be any point of X and $\{x_n\}$ any sequence in X converging to x. We shall denote the set $\{x_n \mid n = 1, 2, \cdots\}$ by A and $K = A \cup \{x\}$. Then K is sequentially compact in X because $x_n \to x$. By hypothesis, f is sequentially compact preserving and hence $f(K)$ is a sequentially compact set of Y. Since $\{f(x_n)\}$ is a sequence in $f(K)$, there exists a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ converging to a point $y \in f(K)$. This implies that f is sequentially subcontinuous.

Remark 2. We have the following implications: continuous $\Rightarrow$ sequentially continuous $\Rightarrow$ sequentially nearly-continuous $\Rightarrow$ sequentially compact preserving $\Rightarrow$ sequentially subcontinuous.

THEOREM 3. *A function $f: X \to Y$ is sequentially compact preserving if and only if $f \mid K: K \to f(K)$ is sequentially subcontinuous for each sequentially compact set K of X.*

Proof.-Necessity. Suppose $f: X \to Y$ is a sequentially compact preserving function. Then $f(K)$ is sequentially compact in Y for each sequentially compact set K of X. Therefore, by (1) of Remark 1, $f \mid K: K \to f(K)$ is sequentially subcontinuous.

Sufficiency. Let K be any sequentially compact set of X and we shall show that $f(K)$ is sequentially compact in Y. Let $\{y_n\}$ be any sequence in $f(K)$. Then, for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $\{x_n\}$ is a sequence in the sequentially compact set K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x \in K$. By hypothesis, $f \mid K : K \to f(K)$ is sequentially subcontinuous and hence there exists a subsequence of $\{y_{n_k}\}$ converging to a point $y \in f(K)$. This implies that $f(K)$ is sequentially compact in Y.

The following corollary gives a sufficient condition for a sequentially subcontinuous function to be sequentially compact preserving.

COROLLARY 1. *If a function $f: X \to Y$ is sequentially subcontinuous and $f(K)$ is sequentially closed in Y for each sequentially compact set K of X, then f is sequentially compact preserving.*

Proof. By Theorem 3, it suffices to prove that $f \mid K : K \to f(K)$ is sequentially subcontinuous for each sequentially compact set K of X. Let $\{x_n\}$ be any sequence in K converging to a point $x \in K$. Then, since f is sequentially subcontinuous, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $y \in Y$ such that $f(x_{n_k}) \to y$. Since $\{f(x_{n_k})\}$ is a sequence in the sequentially closed set $f(K)$ of Y, we obtain $y \in f(K)$. This implies that $f \mid K : K \to f(K)$ is sequentially subcontinuous.

Remark 3. In a semi-Hausdorff space, every sequentially compact set is sequentially closed [4, Theorem 7]. Therefore, the converse of Corollary 1 is also true if Y is semi-Hausdorff.

4. SEQUENTIALLY CLOSED GRAPHS

Let $f: X \to Y$ be a function. The subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and is denoted by $G(f)$. We shall give a sufficient condition for a sequentially subcontinuous function to be sequentially continuous.

THEOREM 4. *If a function $f: X \to Y$ is sequentially subcontinuous and $G(f)$ is sequentially closed, then f is sequentially continuous.*

Proof. Let us assume that f were not sequentially continuous. Then there exist a point $x \in X$ and a sequence $\{x_n\}$ in X such that $x_n \to x$ and the sequence $\{f(x_n)\}$ does not converge to $f(x)$. Since $\{f(x_n)\}$ does not converge to $f(x)$, there exists a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ such that no subsequence of $\{f(x_{n_k})\}$ converges to $f(x)$. Now, since $x_n \to x$, we have $x_{n_k} \to x$. Moreover, f is sequentially subcontinuous and hence there exist a subsequence $\{x_j\}$ of $\{x_{n_k}\}$ and a point $y \in Y$ such that $f(x_j) \to y$. Thus $\{(x_j, f(x_j))\}$ is a sequence in $G(f)$ converging to (x, y). Since $G(f)$ is sequentially closed, we obtain $(x, y) \in G(f)$. Therefore, the subsequence $\{f(x_j)\}$ of $\{f(x_{n_k})\}$ converges to $f(x)$. We have a contradiction.

COROLLARY 2. *Let a function* $f: X \to Y$ *have the sequentially closed graph. Then, the following conditions* (1), (2), (3) *and* (4) *on* f *are equivalent. If also* X *is first countable, then they are equivalent to* (5).

(1) f *is sequentially subcontinuous*;

(2) f *is sequentially compact preserving*;

(3) f *is sequentially nearly-continuous*;

(4) f *is sequentially continuous*;

(5) f *is continuous*.

Proof. This follows immediately from Remark 2, Theorem 4 and [1, 6.3, p. 218].

In [3], P. E. Long showed that if a function of a first countable space into a countably compact space has the closed graph, then it is continuous. We shall give a similar result to this theorem.

COROLLARY 3. *Let* f *be a function of a first countable space* X *into a sequentially compact space* Y. *If* G(f) *is sequentially closed, then* f *is continuous*.

Proof. Since Y is sequentially compact, by (1) of Remark 1, f is sequentially subcontinuous. By hypothesis, G (f) is sequentially closed and hence, by Corollary 2, f is continuous.

REFERENCES

[1] J. DUGUNDJI (1966) – *Topology*, Allyn and Bacon, Boston.

[2] R. V. FULLER (1968) – *Relations among continuous and various non-continuous functions*, « Pacific J. Math. », *25*, 495–509.

[3] P. E. LONG (1969) – *Functions with closed graphs* « Amer. Math. Monthly », *76*, 930–932.

[4] M. G. MURDESHWAR and S. A. NAIMPALLY (1966) – *Semi-Hausdorff spaces*, « Canad. Math. Bull. », *9*, 353–356.