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On involution geometry of a projective line over a field of characteristic 2

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Geometria. — *On involution geometry of a projective line over a field of characteristic 2.* Nota di VINCENZO DICUONZO, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Oggetto di questa Nota è la costruzione di un piano metrico gruppale mediante le involuzioni di una retta proiettiva sopra un campo di caratteristica 2 e gli automorfismi interni del gruppo generato da esse.

INTRODUCTION

In previous works (see references [2] and [3]) some metric projective planes of hyperbolic type were constructed by means of the involutions of a projective line r over a field K of characteristic $\neq 2$. In the present one the same problem is solved when K is of characteristic 2 and order $q > 2$: in this way we obtain a metric affine plane, whose motions are defined as inner automorphisms of the group G of the projectivities on r. We exclude the fundamental field of characteristic 2 since in this case the structure of the group G is very simple.

1. THE ELEMENTS OF THE METRIC PLANE Π

Let r be a *projective line over a field K of characterstic 2 and order $q > 2$.* As it is known the involutions on r are $q^2 - 1$ and parabolic.

Such *involutions* are assumed as *lines of a plane* Π, *whose points are defined as pencils of involutions on r.*

Let us recall that a pencil $\mathscr{P}$ of involutions on r is called hyperbolic, elliptic or parabolic, according as its elements have a common pair of corresponding points or not, or have the same fixed point. This means that the points of Π are of three types: exactly a *point* of Π is called *proper*, *quasiproper* or *singular*, according as the corresponding pencil of involutions is elliptic, hyperbolic or parabolic: it follows that there are $\frac{1}{2}q(q-1)$ proper points, $\frac{1}{2}q(q+1)$ quasiproper points and $(q+1)$ singular points.

A point P of Π is said to belong to a line a of Π, if the pencil corresponding to P contains the involution representing a.

Since a pencil $\mathscr{P}$ of involutions on r contains $q+1$ elements if it is elliptic, and $q-1$ elements if it is hyperbolic or parabolic, *there are $q+1$ lines through a proper point and $q-1$ lines through a quasiproper or singular point.*

Moreover *a line contains $q+1$ points*: exactly one singular point, $q/2$ quasiproper points and $q/2$ proper points. In fact an involution a on r belongs

(*) Nella seduta dell'8 marzo 1975.

obviously to one parabolic pencil, to $q/2$ hyperbolic pencils (as many as the pairs of different points corresponding by the involution *a*) and at last to $q/2$ elliptic pencils (since an elliptic pencil of involutions on r contains $q + 1$ elements and furthermore there are $q^2 - 1$ involutions and $\frac{1}{2} q (q - 1)$ elliptic pencils of involutions on r).

2. THE POINTS OF Π AS ABELIAN SUBGROUPS OF G

Let us observe that the nonparabolic projectivities on r characterize the nonsingular points of Π: in fact such a projectivity is the product of two involutions, which do not commute and hence belong to a nonparabolic pencil.

Since any product of three involutions on r, which are in a nonparabolic pencil $\mathscr{P}$, is equal to an involution of $\mathscr{P}$, the products of pairs of involutions of $\mathscr{P}$ forms an abelian group: this means that *for every nonsingular point of* Π *there is an abelian subgroup S of G, such that the elements of S are even elements of G, that is, products of two generators.*

Also for every singular point of Π there is an abelian subgroup of G. In fact the involutions of a parabolic pencil $\mathscr{P}$ are mutually commuting and moreover the product of any two of them is an involution of $\mathscr{P}$ or the identity. Therefore we can say that *for every singular point of* Π *there is an abelian subgroup of G, whose elements are the identity and the involutions with the same fixed point.*

3. THE QUASIPROPER LINES OF Π.

According to the definition of points of Π, *two lines of* Π *determine a point of* Π, *but* remark that *there can be no line through two distinct points* P_1 and P_2: in fact, any two parabolic pencils of involutions on r have not a common element and moreover a hyperbolic pencil of involutions has not a common element with two parabolic pencils.

In this cases the *points* P_1 and P_2 are called *disjoint.*

In order to obtain the incidence structure of an affine plane we introduce some new lines by the following definition.

It is called *quasiproper line through a singular point* P *the set consisting of* P *and the quasiproper points disjoint from* P.

Obviously there are $q + 1$ quasiproper lines, besides $q^2 - 1$ lines represented by involutions, which will be called *proper lines* from now on.

Moreover there are two quasiproper lines through a quasiproper point and one quasiproper line through a singular point.

Furthermore a quasiproper line contains q quasiproper points, besides a singular point: in fact on r there are q unordered point pairs with a same point.

4. Parallelism and orthogonality on Π

Two lines of Π *are called parallel if they pass through the same singular point.*

Moreover *two proper lines are called orthogonal, if the corresponding involutions commute with each other.*

Since two involutions commute when they belong to a parabolic pencil, *two proper lines, which are parallel, are also orthogonal and vice versa.*

For this reason we extend orthogonality also to the quasiproper lines by the following definition: *any two lines through the same singular point are called orthogonal.* It follows that orthogonality coincides with parallelism, any line may be considered self-orthogonal and moreover also the parabolic pencils are self-orthogonal.

5. The motion group of Π

In order to define the motions of Π we consider the group G of the projectivities on r. Choosing an involution a of G, let us denote by a^* the inner automorphism of G corresponding to a.

Obviously the involutions of G, commuting with a, are invariant by a^*; moreover a^* transforms pencils of involutions onto pencils of the same type and leaves invariant the ones containing a (see [1] n. 2). This means that a^* transforms lines and points of Π onto lines and points of the same type respectively: particularly the proper line a and its points are fixed, while the lines orthogonal to a, are invariant.

Because of these properties we call a^* *line reflection with respect to the proper line a.*

Now we recall that a proper line contains $q/2$ quasiproper points and each of them is disjoint from two singular points.

Since a^* commutes two parabolic pencils disjoint from a hyperbolic pencil containing a (see [1], n. 3), a^* determines an involution in the set of singular points of Π; this agrees with the property that G and the group of its inner automorphisms are isomorphic.

Now we assume as *motions of* Π *the products of line reflections*, that is, the inner automorphisms of G.

Since any product of three involutions, which are in a nonparabolic pencil $\mathcal{P}$ is an involution of $\mathcal{P}$, *any product of the reflections in three proper lines through a nonsingular point* $\mathcal{P}$ *is equal to the reflection in a proper line through* $\mathcal{P}$ *(three symmetries theorem).*

It follows that *the products of two reflections in proper lines through the same nonsingular point* P *form an abelian group.*

According as the point P is proper or quasiproper, any product of the reflections in two proper lines a and b through P, with $a \neq b$, is called *elliptic* or *hyperbolic rotation* respectively.

For the reflections in proper lines through a singular point P, *the product of any two of them is the reflection in a proper line through* P *or the identity*. This depends on the property that the product of two commuting involutions is an involution.

At last we prove that *every motion of* Π *may be represented as product of at most two line reflections*.

Let a, b, c, d four proper lines of Π. Choosing a proper point P, there is a proper line e through P and the point $a \cap b$, such that $e^* a^* b^* = f^*$, that is, $a^* b^* = e^* f^*$. In the same way there is a line g through P and the point $f \cap c$, such that $f^* c^* = g^* h^*$. At last there is a line i through P and the point $h \cap d$, such that $h^* d^* = i^* l^*$. In this way $a^* b^* c^* d^* = e^* f^* c^* d^* = e^* g^* h^* d^* = e^* g^* i^* l^* = m^* l^*$, where $m^* = e^* g^* i^*$, since pass through P.

It follows that any product of five line reflections is equal to the product of three line reflections and hence any motion of Π may be represented as product of at most three line reflections.

Now let a, b, c be three proper lines, not in a pencil, and P a proper point. By the previous procedure, $a^* b^* c^* = d^* e^* f^*$, where d^* and e^* are reflections in proper lines through P. Moreover let g be the line through P and the singular point of f: we obtain $d^* e^* g^* = h^*$, that is, $d^* e^* = h^* g^*$ e, g, i and hence $a^* b^* c^* = h^* g^* f^* = h^* i^*$, where $i^* = g^* f^*$ since $g \perp f$.

Also a line reflection may be represented as product of two line reflections: in fact if a is a proper line, and b any proper line orthogonal to a, $a^* b^* = c^*$, where c^* is the reflection in a proper line $c \perp a, b$ and hence $a^* = c^* b^*$. This means that *every motion of* Π *may be represented as product of two line reflections*.

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