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Description of a class of differential equations with set-valued solutions. Nota II

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Equazioni funzionali. — *Description of a class of differential equations with set-valued solutions.* Nota II di MICHAŁ KISIELEWICZ, presentata (*) dal Corrisp. G. CIMMINO.

RIASSUNTO. — Nella presente Nota II diamo la dimostrazione del teorema di tipo Orlicz concernente equazioni differenziali con soluzioni a valori, che sono insiemi compatti convessi.

INTRODUCTION

In the Note I ([3] of the reference list) we introduced a fundamental complete metric space $(\mathscr{F}, \rho_{\mathscr{F}})$. In this Note we shall consider a metric space $(\mathscr{H}, d)$, where $\mathscr{H} = H \times \mathscr{F}$ and $d = \max(r, \rho_{\mathscr{F}})$. Obviously $(\mathscr{H}, d)$ is a complete metric space too. In § 4 of this Note we shall show that the set $\mathfrak{S}$ of all pairs $(C^{(0)}, \overset{\approx}{F}) \in \mathscr{H}$ such that the integral equation

$$\text{(1)} \qquad X(t) = C^{(0)} + \int_0^t F(\tau, X(\tau))\, d\tau \qquad \text{for} \quad t \in [0, T]\,;\ F \in \overset{\approx}{F}$$

has for every $(C^{(0)}, \overset{\approx}{F}) \in \mathfrak{S}$ more than one solution is of the Baire's first category. The § 3 contains some remarks concerning the uniqueness of the solutions of the integral equation (1).

§ 3. UNIQUENESS OF THE SOLUTIONS OF (1)

It was proved in [1] that if the mapping $F : [0, T] \times H \to H$ satisfies conditions (i)-(iii) of Hypotheses H (F) (see [3]) then the integral equation (1) has for every fixed $(C^{(0)}, \overset{\approx}{F}) \in \mathscr{H}$ at least one solution. Suppose F is also uniformly Lipschitz continuous with respect to $X \in H$. It is easy to see that in this case the integral equation (1) has exactly one solution. Indeed, suppose $X(t)$ and $Y(t)$ are two solutions of (1). Then for every $t \in [0, T]$ we have

$$r(X(t), Y(t)) \leq (1/n) + L \int_0^t r(X(\tau), Y(\tau))\, d\tau\,; \qquad n = 1, 2, \cdots.$$

Hence for $n = 1, 2, \cdots$ and every $t \in [0, T]$ we obtain

$$r(X(t), Y(t)) \leq (1/n) \exp(LT)\,.$$

Therefore $r(X(t), Y(t)) = 0$ for every $t \in [0, T]$.

(*) Nella seduta dell'8 febbraio 1975.

§ 4. THE ORLICZ TYPE THEOREM

Now we shall give our result.

THEOREM 2. *The set* $\mathfrak{S}$ *of those* $(C^{(0)}, \tilde{F}) \in \mathscr{H}$ *for which the integral equation* (1) *has at least two different solutions in* $[0, T]$ *is of the Baire's first category in* $(\mathscr{H}, d)$.

Proof. Let us denote by $\Delta(t, C^{(0)}, \tilde{F})$ the supremum of the numbers $r(X_1(t, C^{(0)}), X_2(t, C^{(0)}))$, where $X_1(t, C^{(0)})$ and $X_2(t, C^{(0)})$ are solutions of (1) corresponding to $(C^{(0)}, \tilde{F}) \in \mathscr{H}$. Let $\{t_\tau\}$ denote a sequence of points of $[0, T]$ dense in $[0, T]$. Then let $\Omega_{MNp\tau}$ denote the set of those $(C^{(0)}, \tilde{F}) \in \mathscr{H}$ for which:

1) $r(C^{(0)}, 0) \leq N$, 2) $\rho_{\mathscr{F}}(\tilde{F}, 0) \leq M$, 3) $\Delta(t_\tau, C^{(0)}, \tilde{F}) \geq 1/p$.

We shall show that $\Omega_{MNp\tau}$ are closed in $(\mathscr{H}, d)$. Suppose $\{(C_n^{(0)}, \tilde{F}_n)\}$ is a sequence of $\Omega_{MNp\tau}$ such that $d[(C_n^{(0)}, \tilde{F}_n), (C^{(0)}, \tilde{F})] \to 0$ as $n \to \infty$. It is easy to see that $C^{(0)}$ and $\tilde{F}$ satisfy 1) and 2) respectively. Furthermore there is a subsequence $\{(C_{nk}^{(0)}, F_{nk})\}$ of $\{(C_n^{(0)}, F_n)\}$ such that $r(C_{nk}^{(0)}, C^{(0)}) \to 0$ and $\sup r(F_{nk}(t, X), F(t, X)) \to 0$ as $k \to \infty$ for almost every $t \in [0, T]$; $F \in \tilde{F}$, $F_{nk} \in \tilde{F}_{nk}$. By 3) there exist functions $X_{nk}^{(1)}(t)$, $X_{nk}^{(2)}(t)$ such that

$$X_{nk}^{(i)}(t) = C_{nk}^{(0)} + \int_0^t F_{nk}(s, X_{nk}^{(i)}(s))\, ds \qquad \text{for} \quad t \in [0, T]$$

and

$$r(X_{nk}^{(1)}(t_\tau), X_{nk}^{(2)}(t_\tau)) \geq 1/p - 1/k. \tag{2}$$

From $\sup_{X \in H} r(F_{nk}(t, X), F(t, X)) \to 0$ it follows the existence of $N(1)$ such that $\sup_{X \in H} r(F_{nk}(t, X), F(t, X)) < 1$ for almost every $t \in [0, T]$ and $k \geq N(1)$. Then for almost every $t \in [0, T]$, every $X \in H$ and $k \geq N(1)$ we have: $r(F_{nk}(t, X), 0) \leq 1 + \mathscr{S}(t)$, where $\mathscr{S}$ is a Lebesgue integrable function such that $r(F(t, X), 0) \leq \mathscr{S}(t)$ for $t \in [0, T]$ and $X \in H$. Taking $\Gamma(t) = \max(1 + \mathscr{S}(t), \mathscr{S}_{n_1}(t), \cdots, \mathscr{S}_{n_{N(1)}}(t))$, where $\mathscr{S}_{n_j}$ is a Lebesgue integrable function on $[0, T]$ such that $r(F_{n_j}(t, X), 0) \leq \mathscr{S}_{n_j}(t)$ $(j = 1, \cdots, N(1))$, we have $r(F_{nk}(t, X), 0) \leq \Gamma(t)$ for $(t, X) \in [0, T] \times H$ and $k = 1, 2, \cdots$. Therefore for every $t, t_1, t_2 \in [0, T]$, $i = 1, 2$ and $k = 1, 2, \cdots$ we have

$$r(X_{nk}^{(i)}(t), 0) \leq r(C_n^{(0)}, 0) + \int_0^t r(F_{nk}(s, X_n^{(i)}(s)), 0)\, ds \leq$$

$$\leq N + \int_0^T \sup_{X \in H} r(F_{nk}(t, X), 0)\, dt \leq N + M$$

and

$$r(X_n^{(i)}(t_1), X_n^{(i)}(t_2)) \leq \left| \int_{t_1}^{t_2} \Gamma(s)\, ds \right| .$$

Hence, by Arzela's theorem there are subsequences $\{X_k^{(i)}(t)\}$ of $\{X_{n_k}^{(i)}(t)\}$ $(i = 1, 2)$ uniformly convergent on $[0, T]$. Let $r(X_k^{(i)}(t), X^{(i)}(t)) \rightrightarrows 0$ as $k \to \infty$ $(i = 1, 2)$. For $t \in [0, T]$ and $i = 1, 2$ we have

$$r\left(X^{(i)}(t)), C^{(0)} + \int_0^t F(s, X^{(i)}(s))\, ds\right) \leq$$

$$\leq r(X^{(i)}(t), X_k^{(i)}(t)) + r\left(C_k^{(0)} + \int_0^t F_k(s, X_k^{(i)}(s))\, ds, C^{(0)} + \int_0^t F(s, X^{(i)}(s))\, ds\right) \leq$$

$$\leq r(X^{(i)}(t), X_k^{(i)}(t)) + r(C_k^{(0)}, C^{(0)}) + \int_0^t r(F_k(s, X_k^{(i)}(s)), F_k(s, X^{(i)}(s))\, ds +$$

$$+ \int_0^t r(F_k(s, X^{(i)}(s)), F(s, X^{(i)}(s)))\, ds$$

for $t \in [0, T]$, $k = 1, 2, \cdots$ and $i = 1, 2$. Hence it is easy to see that $X^{(1)}(t)$ and $X^{(2)}(t)$ are solutions of (1) corresponding to $(C^{(0)}, \tilde{F}) \in \mathscr{H}$. Furthermore from (2) we have $r(X^{(1)}(t), X^{(2)}(t)) \geq 1/p$. Consequently, $(C^{(0)}, \tilde{F}) \in \Omega_{MNp\tau}$. We shall show that $\Omega_{MNp\tau}$ are non-dense in $(\mathscr{H}, d)$. Suppose, on the contrary, that $\Omega_{MNp\tau}$ is dense in a sphere S_h with center $(C_0^{(0)}, \tilde{F}_0) \in \mathscr{H}$ and radius $h > 0$. Then $S_h \subset \overline{\Omega}_{MNp\tau} = \Omega_{MNp\tau}$. By (iv) of Hypotheses H (F) for $F_0 \in \tilde{F}_0$ and every $\delta > 0$ there is a mapping $G_\delta : [0, T] \times H \to H$ such that the conditions (*a*)-(*c*) of (iv) are fulfilled. Then $r(G_\delta(t, X), F_0(t, X)) < \delta$ for every $(t, X) \in [0, T] \times H$. Therefore for $t \in [0, T]$ we have $\sup_{X \in H} r(G_\delta(t, X), F_0(t, X)) \leq \delta$. Taking $\delta < h/T$ we obtain

$$\rho_{\mathscr{F}}(\tilde{G}_\delta, \tilde{F}_0) = \int_0^T \sup_{X \in H} r(G_\delta(t, X), F_0(t, X))\, dt \leq \delta T < h .$$

Therefore $(C_0^{(0)}, \tilde{G}_\delta) \in S_h \subset \Omega_{MNp\tau}$. But $G_\delta \in \tilde{G}_\delta$ is uniformly Lipschitz continuous with respect to $X \in H$, then (1) has for $(C_0^{(0)}, \tilde{G}_\delta)$ exactly one solution. Hence $(C_0^{(0)}, \tilde{G}_\delta) \notin S_h$. Now, our theorem follows from the identity

$$\mathfrak{S} = \bigcup_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{\tau=1}^{\infty} \Omega_{MNp\tau} .$$

Remark. Suppose additionally that F in Theorem 2 is continuous in $t \in [0, T]$. By virtue of Lemma 2 in [2] the integral equation (1) is equivalent to the initial-value problem

$$DX = F(t, X) \quad , \quad X(0) = C^{(0)},$$

where DX denotes the Hukuhara derivative of the mapping X. So in this case Theorem 2 is true for this initial-value problem.

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