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Comparison and Nonoscillation Theorems for Fourth Order Elliptic Systems

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Equazioni a derivate parziali. — *Comparison and Nonoscillation Theorems for Fourth Order Elliptic Systems.* Nota di TAKAŜI KUSANO e NORIO YOSHIDA, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Scopo di questo lavoro è di stabilire alcuni criteri di non-oscillazione nel senso di Kuks [2] e teoremi di confronto del tipo di Sturm per una classe di sistemi ellittici di equazioni a derivate parziali del quarto ordine.

1. INTRODUCTION

The purpose of this paper is twofold. First, we develop nonoscillation criteria in the sense of Kuks [2] for fourth order *linear* elliptic systems of the form

$$\text{(1)} \qquad L[U] \equiv \sum_{i,j,k,l=1}^{n} D_{ij}(A_{ij} A_{kl} D_{kl} U) + 2 B \sum_{k,l=1}^{n} A_{kl} D_{kl} U + CU = 0$$

where A_{ij}, B and C are $m \times m$ matrix functions and U is an $m \times 1$ vector or $m \times m$ matrix function. The criteria generalize recent results of Yoshida [5] for fourth order single elliptic equations. Secondly, we establish a Picone identity for *quasilinear* elliptic systems of the form (1) and then use it to prove Sturmian comparison theorems for such systems. The Picone identity and comparison theorems are extensions of those given by Chan and Young [1] for the fourth order system $\Delta(A\Delta U) + 2 B\Delta U + CU = 0$. We remark that our derivation of Picone's identity is based on a procedure adapted from our earlier paper [3] and is somewhat different from that of Chan and Young.

2. NONOSCILLATION THEOREMS

Consider the linear system (1) in an unbounded domain R in Euclidean n-space E^n. Points in E^n are denoted by $x = (x_1, \cdots, x_n)$, differentiation with respect to x_i by D_i and differentiation with respect to x_i and x_j by D_{ij}, $i, j = 1, \cdots, n$. It is assumed that $A_{ij} (= A_{ji})$ are symmetric $m \times m$ matrix functions of class $C^2(R)$, that B and C are $m \times m$ matrix functions of class C (R) and that the $mn \times mn$ matrix $(A_{ij}(x))$ is positive definite in R. The domain $\mathfrak{D}(\Omega)$ of L relative to a subdomain Ω of R is defined as the set of all $m \times 1$ vector functions $U \in C^4(\Omega) \cap C^2(\overline{\Omega})$.

Following Kuks [2] we say that the system (1) is *nonoscillatory* in R if there exists $r > 0$ such that (1) has no nontrivial ($m \times 1$ vector) solution U

(*) Nella seduta dell'8 marzo del 1975.

satisfying $U = D_i U = 0$, $i = 1, \cdots, n$, on the boundary of any smooth bounded domain contained in $R_r \equiv R \cap \{x : |x| > r\}$.

We also consider the linear vector or matrix differential operator

$$(2) \qquad M[V] \equiv \sum_{i,j,k,l=1}^{n} D_{ij}(G_{ij} G_{kl} D_{kl} V) + 2H \sum_{k,l=1}^{n} G_{kl} D_{kl} V + KV$$

defined in R, where $G_{ij} (= G_{ji})$ are symmetric $m \times m$ matrix functions of class $C^2(R)$, H and K are $m \times m$ matrix functions of class $C(R)$ and HG_{kl}, $k, l = 1, \cdots, n$, are symmetric.

In analogy to the case of single equations [5], the operator M is said to belong to the class $\mathfrak{M}[L; R_r]$ if, for every bounded subdomain Ω of R_r, the functional

$$\int_\Omega \Big[\Big(\sum_{k,l} A_{kl} D_{kl} U\Big)^T \Big(\sum_{k,l} A_{kl} D_{kl} U\Big) - \Big(\sum_{k,l} G_{kl} D_{kl} U\Big)^T \Big(\sum_{k,l} G_{kl} D_{kl} U\Big) +$$
$$+ 2 U^T \sum_{k,l} (BA_{kl} - HG_{kl}) D_{kl} U + U^T (C - K - HH^T) U \Big] dx$$

is nonnegative for all $m \times 1$ vector functions $U \in C^2(\overline{\Omega})$ such that $U = D_i U = 0$ on $\partial\Omega$, $i = 1, \cdots, n$, where "T" denotes the transposed. For example, letting L and M take the special forms $L_1[U] \equiv \Delta(A \Delta U) + CU$ and $M_1[V] \equiv \Delta(G \Delta V) + KV$, respectively, we see that M_1 belongs to the class $\mathfrak{M}[L_1; \Omega]$ if $A - G$ and $C - K$ are both positive semidefinite in Ω.

We shall need the following lemma which ensures the positivity of the quadratic form defined by

$$Q[\xi] \equiv \sum_{i,j=1}^{n} \xi_i^T P_{ij} \xi_j + 2 \xi_{n+1}^T \sum_{i=1}^{n} \Phi_i \xi_i + \xi_{n+1}^T \Psi \xi_{n+1},$$

where ξ_i are $m \times 1$ vectors and P_{ij}, Φ_i and Ψ are $m \times m$ matrix functions of class $C(\Omega)$, $\Omega \subset R$, and P_{ij} and Φ_i are symmetric. The $mn \times mn$ matrix $P = (P_{ij})$ is assumed symmetric and positive definite in Ω. We denote by ξ the $m(n+1) \times 1$ vector whose transposed is $\xi^T = (\xi_1^T, \cdots, \xi_{n+1}^T)$, by ξ' the $mn \times 1$ vector whose transposed is $\xi'^T = (\xi_1^T, \cdots, \xi_n^T)$, and by Φ the $mn \times m$ matrix whose transposed is $\Phi^T = (\Phi_1, \cdots, \Phi_n)$.

LEMMA. *If the matrix* $\Psi - \Phi^T P^{-1} \Phi$ *is positive semidefinite in* Ω, *then, for any nontrivial vector function* $\xi = \xi(x)$ *of class* $C(\Omega)$ *such that*

$$\xi' + P^{-1} \Phi \xi_{n+1} \not\equiv 0 \quad \textit{in} \quad \Omega,$$

$Q[\xi]$ *is nonnegative in* Ω *and is positive at some point of* Ω.

The conclusion follows immediately if we observe that $Q[\xi]$ can be transformed into

$$(3) \qquad Q[\xi] = (\xi' + P^{-1} \Phi \xi_{n+1})^T P (\xi' + P^{-1} \Phi \xi_{n+1}) + \xi_{n+1}^T (\Psi - \Phi^T P^{-1} \Phi) \xi_{n+1}.$$

For the details see Kusano and Yoshida [3].

THEOREM 1. *The system* (1) *is nonoscillatory in* R *if for some* $r > 0$ *there exists an operator* M *defined by* (2) *which belongs to the class* $\mathfrak{M}[L; R_r]$ *and* $m \times m$ *matrix functions* $\Phi, \Psi_1, \cdots, \Psi_n$ *with the following properties*:

(i) $G_{kl}\Phi$, $k, l = 1, \cdots, n$, *and* $\sum_l G_{kl}\Phi\Psi_l$, $k = 1, \cdots, n$, *are symmetric, and the* $mn \times mn$ *matrix* $(G_{kl}\Phi)$ *is positive definite in* R_r;

(ii) $\sum_{k,l} D_{kl}(G_{kl}\Phi) + 2\sum_{k,l} D_k(G_{kl}\Phi)\Psi_l - \Phi^T\Phi + 2H\Phi - K$ *is negative semidefinite in* R_r;

(iii) $\sum_{k,l} G_{kl}\Phi(D_k\Psi_l) + \sum_{k,l} G_{kl}\Phi\Psi_k\Psi_l + \Phi^T\Phi$ *is negative semidefinite in* R_r;

(iv) *for any bounded smooth subdomain* Ω *of* R_r *the relation*

$$\nabla U - \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{bmatrix} U \not\equiv 0 \qquad in \quad \Omega \tag{4}$$

holds for any nontrivial $m \times 1$ *vector function* $U \in \mathfrak{D}(\Omega)$ *such that* $U = D_i U = 0$ *on* $\partial\Omega$, $i = 1, \cdots, n$, *where* ∇U *denotes the* $mn \times 1$ *vector whose transposed is* $((D_1 U)^T, \cdots, (D_n U)^T)$.

Proof. Suppose, on the contrary, that there exists a smooth bounded subdomain Ω of R_r and a nontrivial solution U of (1) such that $U = D_i U = 0$ on $\partial\Omega$. Applying Green's formula and using the hypothesis that $M \in \mathfrak{M}[L; R_r]$ we have

$$\begin{aligned} \int_\Omega U^T L[U]\,dx = \\ = \int_\Omega \Big[\Big(\sum_{k,l} A_{kl} D_{kl} U\Big)^T \Big(\sum_{k,l} A_{kl} D_{kl} U\Big) + \\ + 2U^T B \sum_{k,l} A_{kl} D_{kl} U + U^T C U\Big]\,dx \geqq \\ \geqq \int_\Omega \Big[\Big(\sum_{k,l} G_{kl} D_{kl} U\Big)^T \Big(\sum_{k,l} G_{kl} D_{kl} U\Big) + \\ + 2U^T H \sum_{k,l} G_{kl} D_{kl} U + U^T (K + HH^T) U\Big]\,dx\,. \end{aligned} \tag{5}$$

On the other hand, it is easy to verify that the following identities hold:

$$\begin{aligned} 2\sum_{k,l} D_k(U^T G_{kl}\Phi D_l U) - \sum_{k,l} D_l(U^T D_k(G_{kl}\Phi)U) = \\ = \sum_{k,l}[2U^T G_{kl}\Phi D_{kl} U - U^T D_{kl}(G_{kl}\Phi)U + 2(D_k U^T) G_{kl}\Phi D_l U]\,, \end{aligned} \tag{6}$$

$$(7) \qquad 2\sum_{k,l} D_k(U^T G_{kl}\Phi\Psi_l U) =$$

$$= 2\sum_{k,l}[U^T D_k(G_{kl}\Phi)\Psi_l U + U^T G_{kl}\Phi(D_k\Psi_l)U + 2U^T G_{kl}\Phi\Psi_l D_k U].$$

Note that (6) follows from the symmetry of $\sum_k D_k(G_{kl}\Phi)$, $l = 1, \cdots, n$, and (7) from the symmetry of $\sum_l G_{kl}\Phi\Psi_l$, $k = 1, \cdots, n$. Since the integrals over Ω of the sums on the right hand sides of (6) and (7) vanish, adding these integrals to the last integral of (5), we obtain

$$(8) \qquad 0 = \int_\Omega U^T L[U]\,dx \geqq$$

$$\geqq \int_\Omega \Big[\Big(\sum_{k,l} G_{kl}D_{kl}U + (H^T + \Phi)U\Big)^T \Big(\sum_{k,l} G_{kl}D_{kl}U + (H^T + \Phi)U\Big) +$$

$$+ U^T\Big(K - \sum_{k,l} D_{kl}(G_{kl}\Phi) - 2H\Phi - 2\sum_{k,l} D_k(G_{kl}\Phi)\Psi_l + \Phi^T\Phi\Big)U\Big]\,dx +$$

$$+ 2\int_\Omega \Big[\sum_{k,l}(D_k U^T)G_{kl}\Phi D_l U - 2U^T\sum_{k,l} G_{kl}\Phi\Psi_l D_k U +$$

$$+ U^T\Big(-\sum_{k,l} G_{kl}\Phi D_k\Psi_l - \Phi^T\Phi\Big)U\Big]\,dx.$$

By condition (ii) the first integral on the last side of (8) is nonnegative. To see that the second integral is positive it suffices to apply the above lemma to its integrand, taking into account conditions (i) and (iii). In fact, in view of (3), the integrand is equal to

$$(9) \qquad \theta^T(G_{kl}\Phi)\theta - U^T\Big(\sum_{k,l} G_{kl}\Phi(D_k\Psi_l) + \sum_{k,l} G_{kl}\Phi\Psi_k\Psi_l + \Phi^T\Phi\Big)U,$$

where θ stands for the left member of (4). Thus the relation (8) leads to a contradiction and the proof is complete.

DEFINITION. Following Chan and Young [1] we say that an $m \times m$ matrix function W of class C^2 is M-*prepared* if the matrices

$$(10) \qquad W^T\sum_{i,j} G_{kl}G_{ij}(D_{ij}W), \qquad k, l = 1, \cdots, n,$$

$$(11) \qquad \sum_{i,j,l}(D_l W^T)G_{kl}G_{ij}(D_{ij}W), \qquad k = 1, \cdots, n,$$

are all symmetric. This definition also applies to the case when M is quasilinear.

THEOREM 2. *The system* (1) *is nonoscillatory in* R *if for some* $r > 0$ *there exists an operator* M *of the type* (2) *which belongs to the class* $\mathfrak{M}[L ; R_r]$ *and an* $m \times m$ *nonsingular matrix function* W *of class* $C^4(R_r)$ *with the following properties*:

(i) W *is an* M-*prepared matrix*;

(ii) *the* $mn \times mn$ *matrix* $\left(\sum_{i,j} G_{kl} G_{ij} (D_{ij} W) W^{-1}\right)$ *is negative definite in* R_r;

(iii) $M[W] W^{-1}$ *is positive semidefinite in* R_r.

Proof. Define the $m \times m$ matrices Φ and Ψ_l by

$$\Phi = -\sum_{i,j} G_{ij} (D_{ij} W) W^{-1} \qquad \text{and} \quad \Psi_l = (D_l W) W^{-1} \qquad l = 1, \cdots, n.$$

That $G_{kl} \Phi$ and $\sum_l G_{kl} \Phi \Psi_l$ are symmetric follows readily from the symmetry of (10) and (11), respectively. Hypothesis (ii) implies that $(G_{kl} \Phi)$ is positive definite. It is a matter of simple calculation to show that

$$\sum_{k,l} D_{kl} (G_{kl} \Phi) + 2 \sum_{k,l} D_k (G_{kl} \Phi) \Psi_l - \Phi^T \Phi + 2 H\Phi - K = -M[W] W^{-1},$$

$$\sum_{k,l} G_{kl} \Phi (D_k \Psi_l) + \sum_{k,l} G_{kl} \Phi \Psi_k \Psi_l + \Phi^T \Phi = 0.$$

Finally, let Ω be any smooth bounded subdomain of R_r and let U be any nontrivial vector of class $\mathfrak{D}(\Omega)$ such that $U = D_i U = 0$ on $\partial\Omega$, $i = 1, \cdots, n$. An easy computation yields

$$\nabla U - \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{bmatrix} U = \begin{bmatrix} WD_1 (W^{-1} U) \\ \vdots \\ WD_n (W^{-1} U) \end{bmatrix}$$

which does not vanish identically in Ω. Now the conclusion follows from Theorem 1. This completes the proof.

Remark. From the above proof we see that the conclusion of Theorem 2 remains valid if the hypotheses (ii) and (iii) are replaced by the following:

(ii′) the $mn \times mn$ matrix (G_{kl}) is positive definite in R_r and the identity

$$-\sum_{i,j} G_{ij} (D_{ij} W) W^{-1} = \varphi(x) I_m \quad , \quad \varphi(x) \geqq 0 \qquad \text{in} \quad R_r$$

holds, where I_m is the $m \times m$ identity matrix;

(iii′) $M[W] W^{-1}$ is positive definite in R_r.

COROLLARY. *The system*

$$L_1[U] \equiv \Delta(A(x)\,\Delta U) + C(x)\,U = 0 \tag{12}$$

is nonoscillatory in R *if the matrices* $A(x)$ *and* $C(x)$ *are uniformly positive definite in* R_r.

Proof. Let $M_1[V] \equiv a_0 \Delta^2 V + c_0 V$, where a_0 and c_0 are positive constants such that $\xi^T A(x)\,\xi \geqq a_0 |\xi|^2$ and $\xi^T C(x)\,\xi \geqq c_0 |\xi|^2$ for all real $m \times 1$ vectors ξ. Then, clearly M_1 belongs to the class $\mathfrak{M}[L_1; R_r]$. Defining $W = |x|^p I_m$, it is easily seen that the number p can be chosen so that the matrix W satisfies the conditions (i)-(iii) of Theorem 2 or the conditions (i), (ii′) and (iii′). (See Yoshida [5]). It follows that the system (12) is nonoscillatory.

3. A Picone identity and comparison theorems

Let Ω be a smooth bounded domain in E^n and let Y be a domain in E^m containing the origin. Assume that L is a quasilinear vector differential operator of the type (1). The coefficients $A_{ij}(x, \eta)$ are $m \times m$ symmetric matrix functions of class $C^2(\overline{\Omega} \times Y)$, $B(x, \eta)$ and $C(x, \eta)$ are $m \times m$ matrix functions of class $C(\overline{\Omega} \times Y)$, and the $mn \times mn$ matrix $(A_{ij}(x, \eta))$ is positive definite in $\Omega \times Y$. The domain $\mathfrak{D}$ of L is defined as the set of all $m \times 1$ vector functions of class $C^4(\Omega) \cap C^2(\overline{\Omega})$ with range in Y. Assume further M is a quasilinear matrix differential operator of the type (2). It is assumed that $G_{ij}(x, \zeta)\ (= G_{ji}(x, \zeta))$ are $m \times m$ symmetric matrix functions of class $C^2(\overline{\Omega} \times Y^m)$, that $H(x, \zeta)$ and $K(x, \zeta)$ are $m \times m$ matrix functions of class $C(\overline{\Omega} \times Y^m)$ and that HG_{kl} are symmetric. The domain of M is the set $\mathfrak{D}^m$ of all $m \times m$ matrix functions W whose column vectors W_i, $i = 1, \cdots, m$, belong to $\mathfrak{D}$.

Now let $W \in \mathfrak{D}^m$ be a nonsingular M-prepared matrix and let $\Phi \in C^2(\overline{\Omega} \times Y^m)$, $\Psi_l \in C^1(\overline{\Omega} \times Y^m)$, $l = 1, \cdots, n$, be $m \times m$ matrices such that $G_{kl}\Phi$ and $\sum_l G_{kl}\Phi\Psi_l$ are symmetric and $(G_{kl}\Phi)$ is positive definite in $\Omega \times Y^m$. Using the identities (6) and (7) which also hold for the quasilinear case, we obtain

$$\Big(\sum_{k,l} G_{kl} D_{kl} U\Big)^T \Big(\sum_{k,l} G_{kl} D_{kl} U\Big) +$$

$$+ 2\,U^T H \sum_{k,l} G_{kl} D_{kl} U + U^T (K + \Gamma)\,U =$$

$$= \Big(\sum_{k,l} G_{kl} D_{kl} U + (H^T + \Phi)\,U\Big)^T \Big(\sum_{k,l} G_{kl} D_{kl} U + (H^T + \Phi)\,U\Big) +$$

$$+ U^T (\Gamma - HH^T)\,U + U^T \Big(K - \sum_{k,l} D_{kl}(G_{kl}\Phi) -$$

$$- 2\,H\Phi - 2\sum_{k,l} D_k\,(G_{kl}\,\Phi)\,\Psi_l + \Phi^T\,\Phi\Big)\,U +$$

$$+ 2\Big[\sum_{k,l} (D_k\,U^T)\,G_{kl}\,\Phi D_l\,U - 2\,U^T\sum_{k,l} G_{kl}\,\Phi\Psi_l\,D_k\,U +$$

$$+ U^T\Big(-\sum_{k,l} G_{kl}\,\Phi D_k\,\Psi_l - \Phi^T\,\Phi\Big)\,U\Big] -$$

$$- 2\sum_{k,l} D_k\,(U^T\,G_{kl}\,\Phi D_l\,U) + \sum_{k,l} D_l\,(U^T\,D_k\,(G_{kl}\,\Phi)\,U) +$$

$$+ 2\sum_{k,l} D_k\,(U^T\,G_{kl}\,\Phi\Psi_l\,U)\,,$$

where Γ is an arbitrary $m\times m$ matrix function defined in $\overline{\Omega}\times Y^m$. Substituting

$$\Phi = -\sum_{i,j} G_{ij}\,(D_{ij}\,W)\,W^{-1}\;,\qquad \Psi_l = (D_l\,W)\,W^{-1},\qquad l = 1\,,\cdots, n,$$

into the right-hand side of the above and recalling the transformation which led to (9), we have

$$(13)\qquad \Big(\sum_{k,l} G_{kl}\,D_{kl}\,U\Big)^T\Big(\sum_{k,l} G_{kl}\,D_{kl}\,U\Big) +$$

$$+ 2\,U^T\,H\sum_{k,l} G_{kl}\,D_{kl}\,U + U^T\,(K+\Gamma)\,U =$$

$$= \Big(\sum_{k,l} G_{kl}\,(D_{kl}\,U - (D_{kl}\,W)\,W^{-1}\,U) + H^T\,U\Big)^T\cdot$$

$$\cdot\Big(\sum_{k,l} G_{kl}\,(D_{kl}\,U - (D_{kl}\,W)\,W^{-1}\,U) + H^T\,U\Big) +$$

$$+ U^T\,(\Gamma - HH^T)\,U + U^T\,M\,[W]\,W^{-1}\,U -$$

$$- 2\sum_{i,j,k,l} (WD_k\,(W^{-1}\,U))^T\,G_{kl}\,G_{ij}\,(D_{ij}\,W)\,W^{-1}\,(WD_l\,(W^{-1}\,U)) +$$

$$+ 2\sum_{i,j,k,l} D_k\,(U^T\,G_{kl}\,G_{ij}\,(D_{ij}\,W)\,W^{-1}\,D_l\,U) -$$

$$- \sum_{i,j,k,l} D_l\,(U^T\,D_k\,(G_{kl}\,G_{ij}\,(D_{ij}\,W)\,W^{-1})\,U) -$$

$$- 2\sum_{i,j,k,l} D_k\,(U^T\,G_{kl}\,G_{ij}\,(D_{ij}\,W)\,W^{-1}\,(D_l\,W)\,W^{-1}\,U)\,.$$

On the other hand, the following identity holds:

$$(14)\qquad U^T\,L\,[U] =$$

$$= \Big(\sum_{k,l} A_{kl}\,D_{kl}\,U\Big)^T\Big(\sum_{k,l} A_{kl}\,D_{kl}\,U\Big) + 2\,U^T\,B\sum_{k,l} A_{kl}\,D_{kl}\,U + U^T\,CU +$$

$$+ \sum_{i,j,k,l} D_i\,(U^T\,D_j\,(A_{ij}\,A_{kl}\,D_{kl}\,U)) - \sum_{i,j,k,l} D_j\,((D_i\,U^T)\,(A_{ij}\,A_{kl}\,D_{kl}\,U))\,.$$

Combining (13) with (14) yields the desired Picone identity:

$$\begin{aligned}
&U^T L[U] - U^T M[W] W^{-1} U - \sum_{i,j,k,l} D_i (U^T D_j (A_{ij} A_{kl} D_{kl} U)) \\
&+ \sum_{i,j,k,l} D_j ((D_i U^T)(A_{ij} A_{kl} D_{kl} U)) - \\
&- \sum_{i,j,k,l} D_k (U^T G_{kl} G_{ij} (D_{ij} W) W^{-1} D_l U) - \\
&- \sum_{i,j,k,l} D_k (U^T G_{kl} G_{ij} (D_{ij} W) D_l (W^{-1} U)) + \\
&+ \sum_{i,j,k,l} D_l (U^T D_k (G_{kl} G_{ij} (D_{ij} W)) W^{-1} U) = \\
&= \left(\sum_{k,l} A_{kl} D_{kl} U\right)^T \left(\sum_{k,l} A_{kl} D_{kl} U\right) + 2 U^T B \sum_{k,l} A_{kl} D_{kl} U + U^T C U - \\
&- \left(\sum_{k,l} G_{kl} D_{kl} U\right)^T \left(\sum_{k,l} G_{kl} D_{kl} U\right) - \\
&- 2 U^T H \sum_{k,l} G_{kl} D_{kl} U - U^T (K + \Gamma) U - \\
&- 2 \sum_{i,j,k,l} (W D_k (W^{-1} U))^T G_{kl} G_{ij} (D_{ij} W) W^{-1} (W D_l (W^{-1} U)) \\
&+ \left(\sum_{k,l} G_{kl} (D_{kl} U - (D_{kl} W) W^{-1} U) + H^T U\right)^T \cdot \\
&\cdot \left(\sum_{k,l} G_{kl} (D_{kl} U - (D_{kl} W) W^{-1} U) + H^T U\right) + U^T (\Gamma - H H^T) U .
\end{aligned}$$

An integration of the above identity with the use of the divergence theorem would give the Picone identity in integral form which extends the one formulated in Theorem 1 of Chan and Young [1]. Here we present the following two specialized versions of Picone's integral identity in terms of the functionals defined by

$$\begin{aligned}
f[U] = \int_\Omega \Bigg[&\left(\sum_{k,l} A_{kl} D_{kl} U\right)^T \left(\sum_{k,l} A_{kl} D_{kl} U\right) + \\
&+ 2 U^T B \sum_{k,l} A_{kl} D_{kl} U + U^T C U \Bigg] dx ,
\end{aligned}$$

$$\begin{aligned}
F[U, W] = \int_\Omega \Bigg[&\left(\sum_{k,l} G_{kl} D_{kl} U\right)^T \left(\sum_{k,l} G_{kl} D_{kl} U\right) + \\
&+ 2 U^T H \sum_{k,l} G_{kl} D_{kl} U + U^T (K + \Gamma) U \Bigg] dx ,
\end{aligned}$$

$$J[U, W] = -2\int_{\Omega} \Big[\sum_{i,j,k,l} (WD_k(W^{-1}U))^T \cdot$$

$$\cdot G_{kl} G_{ij} (D_{ij} W) W^{-1} (WD_l(W^{-1}U))\Big] dx,$$

$$Q[U, W] = \int_{\Omega} \Big[\Big(\sum_{k,l} G_{kl}(D_{kl}U - (D_{kl}W) W^{-1}U) + H^T U\Big)^T \cdot$$

$$\cdot \Big(\sum_{k,l} G_{kl}(D_{kl}U - (D_{kl}W) W^{-1}U) + H^T U\Big) + U^T(\Gamma - HH^T)U\Big] dx.$$

THEOREM 3. *Let* $\partial\Omega$ *be piecewise smooth and let* U *be an* $m \times 1$ *vector function of class* $C^2(\overline{\Omega})$ *with range in* Y *and such that* $U = 0$ *on* $\partial\Omega$. *Then, every* M-*prepared matrix* $W \in \mathfrak{D}^m$ *such that* $W^{-1}U \in C^2(\overline{\Omega})$ *with range in* Y *satisfies*

$$F[U, W] = \int_{\Omega} U^T M[W] W^{-1} U \, dx + J[U, W] + Q[U, W].$$

THEOREM 4. *Let* $\partial\Omega$ *be piecewise smooth and let* U *be an* $m \times 1$ *vector function of class* $\mathfrak{D}$ *and such that* $U = D_i U = 0$ *on* $\partial\Omega$, $i = 1, \cdots, n$. *Then, every* M-*prepared matrix* $W \in \mathfrak{D}^m$ *such that* $W^{-1}U \in C^2(\overline{\Omega})$ *with range in* Y *satisfies*

$$\int_{\Omega} [U^T L[U] - U^T M[W] W^{-1} U] \, dx =$$

$$= f[U] - F[U, W] + J[U, W] + Q[U, W].$$

Once the Picone identity has been established, it is not difficult to use it to derive various types of Sturmian comparison theorems for the system (1) as given in Chan and Young [1]. Here we only state results which correspond to Theorems 2 and 4 of Chan and Young.

THEOREM 5. *Let* $\partial\Omega$ *be piecewise smooth and suppose that*

(i) *the* $m \times m$ *matrix* $\Gamma = \Gamma(x, \zeta)$ *is such that* $\Gamma - HH^T$ *is positive definite in* $\Omega \times Y^m$;

(ii) W *is an* M-*prepared matrix of class* $\mathfrak{D}^m$ *such that* $W^T M[W]$ *and the* $mn \times mn$ *matrix* $\Big(-W^T G_{kl} \sum_{i,j} G_{ij} D_{ij} W\Big)$ *are positive semidefinite in* $\Omega \times Y^m$;

(iii) *there exists a nontrivial* $U \in C^2(\overline{\Omega})$ *with range in* Y *such that* $U = 0$ *on* $\partial\Omega$ *and* $F[U, W] \leqq 0$.

Then, det W *vanishes at some point of* $\overline{\Omega}$.

THEOREM 6. *Let* $\partial\Omega$ *be of class* C^2 *and suppose that*

(i) *the* $m \times m$ *matrix* $\Gamma = \Gamma(x, \zeta)$ *is such that* $\Gamma - HH^T$ *is positive definite in* $\Omega \times Y^m$;

(ii) W *is an* M-*prepared matrix of class* $\mathfrak{D}^m$ *such that* $W^T M[W]$ *and the* $mn \times mn$ *matrix* $\left(-W^T G_{kl} \sum_{i,j} G_{ij} D_{ij} W\right)$ *are positive semidefinite in* $\Omega \times Y^m$;

(iii) *there exists a nontrivial* $U \in \mathfrak{D}$ *such that*

$$U = D_i U = 0 \quad on \quad \partial\Omega, \qquad i = 1, \cdots, n,$$

$$\int_\Omega U^T L[U]\, dx \leqq 0,$$

$$f[U] \geqq F[U, W].$$

Then, det W *vanishes at some point of* Ω.

Theorem 5 follows immediately from Theorem 3. Theorem 6 can be proved with the use of Theorem 4 and on the basis of an approximation argument which goes back to Swanson [4]. We omit the proofs of these theorems, as they are almost duplications of those presented by Chan and Young.

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