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Further remarks on the existence of periodic solutions of certain fifth order non-linear differential equations

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Equazioni differenziali ordinarie. — *Further remarks on the existence of periodic solutions of certain fifth order non-linear differential equations.* Nota di JAMES O. C. EZEILO e HAROON O. TEJUMOLA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si danno condizioni più generali di quelle contenute in una ricerca precedente per l'esistenza di almeno una soluzione periodica delle due equazioni:

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + \Phi(\dot{x}) \ddot{x} + a_4 \dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) = p(t)$$

$$x^{(5)} + f(\dddot{x}) x^{(4)} + a_2 \dddot{x} + a_4 \dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) = p(t),$$

$$p(t + \omega) = p(t), \omega > 0.$$

1. In a previous paper [1] we examined the two fifth order differential equations:

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + \varphi(\dot{x}) \ddot{x} + a_4 \dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) = p(t) \tag{1.1}$$

$$x^{(5)} + f(\ddot{x}) x^{(4)} + a_2 \dddot{x} + a_3 \ddot{x} + a_4 \dot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) = p(t) \tag{1.2}$$

in which a_1, a_2, a_3, a_4 are positive constants and f, φ, h, p are continuous functions of the arguments shown, with h bounded and $p(t)$ ω-periodic in t (that is $p(t + \omega) = p(t)$) for some $\omega > 0$. We showed there, subject to the further conditions:

$$h(x, y, z, u, v) \operatorname{sgn} x > 0 \ (|x| \geq x_0) \qquad \text{and} \qquad \left| \int_0^t p(s)\, ds \right| \leq B$$

(B constant) on h and p, that

(I) *if there exist positive constants* a_3, β_0, γ_0 *such that*

$$\varphi(y) \geq a_3 \qquad \textit{and} \quad \{a_1 a_2 \varphi(y)\} a_3 - a_1^2 a_4 \geq \beta_0, \tag{1.3}$$

$$\textit{for} \quad |y| \geq \gamma_0 \qquad \textit{then} \quad (1.1)$$

has at least one ω*-periodic solution, and that*

(II) *if there exist positive constants* a_1, β_1, ζ_0 *such that*

$$f(z) \geq a_1 \qquad \textit{and} \quad (a_1 a_2 - a_3) a_3 - a_1 a_4 f(z) \geq \beta_1 \tag{1.4}$$

$$\textit{for} \quad |z| \geq \zeta_0$$

then (1.2) *also has at least one* ω*-periodic solution*. Our object in the present note is to further improve these results (I) and (II) by relaxing the condit-

(*) Nella seduta dell'8 marzo 1975.

ions (1.3) and (1.4) to the following:

(1.5) $\varphi(y) > 0$ and $\{a_1 a_2 - \varphi(y)\}\, \varphi(y) - a_1^2 a_4 \geq \beta_0$ for $|y| \geq \gamma_0$

(1.6) $f(z) > 0$ and $(a_2 f(z) - a_3)\, a_3 - a_4 f^2(z) \geq \beta_1$ for $|z| \geq \zeta_0$

respectively.

It is readily checked, by taking the difference between

$$\{(a_1 a_2 - \varphi)\, a_3 - a_1^2 a_4\} \quad \text{and} \quad \{(a_1 a_2 - \varphi)\, \varphi - a_1^2 a_4\}$$

and between

$$\{(a_1 a_2 - a_3)\, a_3 - a_1 a_4 f\} \quad \text{and} \quad \{(a_2 f - a_1)\, a_3 - a_4 f^2\}$$

that (1.3) and (1.4) imply (1.5) and (1.6) respectively. That (1.5) does not, however, imply (1.3) is best illustrated by considering the equation

$$x^{(5)} + x^{(4)} + 4\dddot{x} + (2 - \sin(\dot{x})^3)\, \ddot{x} + 2\dot{x} + h = p .$$

Here $a_1 = 1$, $a_2 = 4$, $a_4 = 2$ and $\varphi(y) = 2 - \sin(y^3)$ so that $a_3 = \inf\limits_{|y| \geq \gamma_0} \varphi(y) = 1$ (*any* $\gamma_0 \geq 0$). Thus $(a_1 a_2 - \varphi)\, a_3 - a_1^2 a_4 \equiv \sin(y^3)$ so that the second condition cannot be satisfied for any $\beta_0 > 0$, whereas, at the same time

$$(a_1 a_2 - \varphi)\, \varphi - a_1^2 a_4 \equiv 2 - \sin^2(y^3) \geq 1$$

for all y, so that (1.5) holds. By considering the equation

$$x^{(5)} + (2 - \sin \dddot{x})\, x^{(4)} + 4\dddot{x} + \ddot{x} + \dot{x} + h = p$$

it can be verified also that (1.6) does not in general imply (1.4).

2. We first tackle equation (1.1), with φ now subject to (1.5).

The proof is by the Leray-Schauder fixed point technique as outlined in [1: § 2.1]. Indeed we found that the rest of our previous treatment of (1.1) in [1] (that is after a misprint which occurs in each of the results (23), (25), (28), (43) and (48) of [1] has been rectified by replacing $\varphi_\mu(y)$ by the capital $\Phi_\mu(y) \equiv \int_0^y \varphi_\mu(s)\, ds$) still holds good under the new hypothesis (1.5) if the constant a_3 which is used in defining the parameter dependent equation (7) of [1] is replaced by a constant $A_3 > 0$ fixed such that

$$(a_1 a_2 - A_3)\, A_3 - a_1^2 a_4 > 0 . \tag{2.1}$$

That such an A_3 can be fixed is easily seen from the two identities:

$$(a_1 a_2 - A_3)\, A_3 - a_1^2 a_4 \equiv -\left(A_3 - \frac{1}{2} a_1 a_2\right)^2 + \frac{1}{4} a_1^2 (a_2^2 - 4 a_4) ,$$

$$(a_1 a_2 - \varphi)\, \varphi - a_1^2 a_4 \equiv -\left(\varphi - \frac{1}{2} a_1 a_2\right)^2 + \frac{1}{4} a_1^2 (a_2^2 - 4 a_4) ,$$

the latter of which shows, by (1.5), that $a_2^2 - 4a_4 > 0$ which, when combined with the former identity, shows that (2.1) can be secured under the present conditions if $\left(A_3 - \frac{1}{2} a_1 a_2\right)$ is sufficiently small.

In order to substantiate the claim above about A_3 being a satisfactory replacement for a_3 in the conversion of our methods in [1] to the present case, only two points need to be verified, namely (i) that a positive constant $b_5 > 0$ can be chosen such that the equation

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + A_3 \ddot{x} + a_4 \dot{x} + b_5 x = 0$$

is asymptotically stable, and (ii) that the function V defined by equation (27) of [1], with a_3 replaced by A_3, retains the same basic Lyapunov properties as before. The first point here is easily disposed of using (2.1) and the positiveness of a_1, a_2, A_3 and a_4 exactly as in [1; § 2.3]. Coming to the second point it is useful to note that the function V, given in [1; § 2.5], is a combination of three function V_1, V_2 and V_3 of which only one component, namely V_1 involves a_3. Indeed, if $V_{1,A}$ denotes V_1 after a_3 has been replaced by A_3, it can be checked readily that

$$V_{1,A} = Q + a_1 R \tag{2.2}$$

where Q is the same quadratic positive definite form which we had in [1], and R is given by

$$2R = 2\mu \int_0^y \{\Phi(s) - a_1 a_1^{-2} a_4 s\}\, ds + (1 - \mu)(A_3 - a_1 a_2^{-1} a_3)\, y^2$$

where $\Phi(s) \equiv \int_0^s \varphi(t)\, dt$. But, by [2; Lemma 1 (III)]

$$\int_0^y \{\Phi(s) - a_1 a_2^{-1} a_4 s\}\, ds \geq -\delta_0$$

for some finite constant $\delta_0 > 0$, and also, quite clearly, $(A_3 - a_1 a_2^{-1} a_4)\, y^2 \geq 0$ since $a_2 A_3 - a_1 a_4 > 0$, by (2.1). Hence,

$$V_{1,A} \geq Q - a_1 \delta_0 \qquad (0 \leq \mu \leq 1).$$

just as in [1] so that one of the two Lyapunov properties used in [1] is also valid here for V when a_3 is replaced by A_3.

The other Lyapunov property arises in connection with the solution of the system (23) of [1]. For ease of reference in what follows let $(23)_A$ denote the system (23) of [1] with a_3 replaced by A_3. To establish the remaining Lyapunov property it will clearly be enough to verify that there are constants $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ such that

$$(2.3)\quad \dot{V}_{1,A} \equiv \frac{d}{dt} V_{1,A}(y, z, u, v) \leq -\delta_1 (y^2 + u^2) + \delta_2 M (|y| + |z| + |v| + \delta_3)$$

for all solutions (y, z, u, v) of $(23)_A$, since the estimates of the derivatives (in the usual notation) $\dot{V}_2^*, \dot{V}_3^*$ along solution paths of $(23)_A$ are much the same as before. Now, if (y, z, u, v) is any solution of $(23)_A$ we have by an elementary calculation from (2.2) that

$$(2.4)\qquad \dot{V}_{1,A} = -U_1 + U_2$$

where

$$U_1 = a_1 u^2 + 2 u\Phi_\mu(y) + a_1 y \Phi_\mu(y) - a_1 a_4 y^2$$

$$U_2 = \{a_2 a_4^{-1} V + (a_2^2 a_4^{-1} - 2) z - a_1 y\} \chi_\mu$$

Here $\varphi_\mu(y) \equiv (1 - \mu) A_3 y + \mu\varphi(y)$ and χ_μ, as in [1], satisfies $|\chi_\mu| \leq M$ so that in particular

$$|U_2| \leq \delta (|v| + |z| + |y|) M$$

for some $\delta > 0$. The function W_1 can be reset thus:

$$U_1 = \mu \{a_1^{-1} [a_1 u + \Phi(y)]^2 + a_1^{-1} [a_1 a_2 y \Phi(y) - \Phi^2(y) - a_1^2 a_4 y]^2\} +$$
$$+ (1 - \mu) \{a_1^{-1} [a_1 u + A_3 y]^2 + a_1^{-1} [a_1 a_2 A_3 - A_3^2 - a_1^{-1} a_4] y^2\}$$

by use of the actual definition of $\Phi_\mu(y)$. The expression inside the first pair of brace brackets here is, except for an obvious difference in notation, the same as the expression W_1 given by equation (6.5) of [2], and the estimates there in [2; § 6] show that here

$$a_1^{-1} [a_1 u + \Phi(y)]^2 + a_1^{-1} [a_1 a_2 y \Phi(y) - \Phi^2(y) - a_1^2 a_4 y^2] \geq \delta_3 (y^2 + u^2) - \delta_4$$

for some constants $\delta_3 > 0$, $\delta_4 > 0$. Next, by (2.1), the expression inside the second pair of brace brackets satisfies

$$a_1^{-1} [a_1 u + A_3 y]^2 + a_1^{-1} [a_1 a_2 A_3 - A_3^2 - a_1^2 a_4] y^2 \geq \delta_5 (y^2 + u^2)$$

for some $\delta_5 > 0$. Thus, for $0 \leq \mu \leq 1$,

$$U_1 \geq \delta_6 (y^2 + u^2) - \delta_4$$

where $\delta_6 = \min(\delta_3, \delta_5)$, and (2.3) now follows on combining the estimates of U_1, U_2 with (2.4).

This concludes the verification of the essential properties of the constant A_3 defined by (2.1), and the existence of an ω-periodic solution of (1.1) with φ subject to (1.5) can then follow as before.

3. We consider next the equation (1.2) with f now subject to (1.6). The procedure is essentially as in [1] and we shall thus sketch only the outlines here. The starting point, then, is the special parameter dependent equation:

$$x^{(5)} + f_\mu(\ddot{x})\, x^{(4)} + a_2\, \dddot{x} + a_3\, \ddot{x} + a_2\, \dot{x} + h_\mu^* = \mu p \qquad (0 \leq \mu \leq 1) \tag{3.1}$$

with h_μ^* as before (see (11) of [1]) and f_μ defined by

$$f_\mu = (1 - \mu)\, A_1 + \mu f(\ddot{x})$$

where $A_1 > 0$ is a constant fixed such that

$$(A_1\, a_2 - a_3)\, a_3 - A_1^2\, a_4 > 0 \tag{3.2}$$

That such a constant can be determined follows in the same way as before from (1.6). Because of (3.2) the linear system corresponding to $\mu = 0$ in (3.1) is asymptotically stable if $b_5 > 0$ is sufficiently small and it remains then to establish the requisite boundedness results for (3.1).

The ultimate boundedness of $|\dot{x}(t)|$, $|\ddot{x}(t)|$, $|\dddot{x}(t)|$ and $|x^{(4)}(t)|$ for any solution $x(t)$ of (3.1) is again best tackled, separately, by considering fourth order equation.

$$y^{(4)} + f_\mu(\dot{y})\, \dddot{y} + a_2\, \ddot{y} + a_3\, y + a_4\, y = \chi_\mu \tag{3.3}$$

obtainable from (3.1) on setting $y = \dot{x}$. The function χ_μ here is as in [1] and is bounded. Thus the boundedness techniques developed in [3] (see particularly §§ 9–11) are applicable and can indeed be used in the same way as before to obtain the ultimate boundedness of $|y|$, $|\dot{y}|$, $|\ddot{y}|$ and $|\dddot{y}|$ for any solution y of (3.3) which is the same thing as the ultimate boundedness of $|\dot{x}(t)|$, $|\ddot{x}(t)|$, $|\dddot{x}(t)|$ and $|x^{(4)}(t)|$ for any solution $x(t)$ of (3.1)

Once the ultimate boundedness of $|\dot{x}(t)|$, $|\ddot{x}(t)|$, $|\dddot{x}(t)|$ and $|x^{(4)}(t)|$ has been established that of $|x(t)|$ can be derived exactly as prescribed in [1, § 3.1]. The required existence result for (1.2) subject to the restriction (1.6) on f then follows.

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