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On the Oscillation of Bounded Solutions of Forced Differential Equations with Deviating Arguments

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Equazioni differenziali ordinarie. — *On the Oscillation of Bounded Solutions of Forced Differential Equations with Deviating Arguments.* Nota di VASILIOS STAIKOS e YIANNIS SFICAS, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori studiano il comportamento oscillatorio e asintotico delle soluzioni dell'equazione differenziale con argomenti ritardati e non autonoma $[r(t)\, x^{(n-m)}(t)]^{(m)} + a(t) f(x[g_1(t)], \cdots, x[g_l(t)]) = b(t)$; il risultato ottenuto estende in varie direzioni uno recente dovuto a T. Kusano e H. Onose. Inoltre discutono ulteriori estensioni dei loro risultati.

Recently there has been an increasing interest in the study of the oscillatory behavior of forced differential equations; however, we may say that, in comparison with the unforced equations, only a few things are known on the subject. For general interest we refer to [1-9] and [11].

This paper is concerned with the oscillatory behavior of bounded solutions of the following differential equation with deviating arguments

$$(*) \qquad [r(t)\, x^{(n-m)}(t)]^{(m)} + a(t) f(x[g_1(t)], x[g_2(t)], \cdots, x[g_l(t)]) = b(t)$$

where l, m, n are positive integers with $n \geqq 2$ and $1 \leqq m \leqq n-1$. Moreover we suppose the following:

(i) a, b are continuous real-valued functions on $[t_0, \infty)$.

(ii) r is continuous and positive function on $[t_0, \infty)$ such that

$$\int^{\infty} \frac{dt}{r(t)} = \infty$$

(iii) f is continuous real-valued function on $\mathbf{R}^l$ such that

$$(\forall i = 1, 2, \cdots, l)\, y_i > 0 \Rightarrow f(y_1, y_2, \cdots, y_l) > 0$$

and

$$(\forall i = 1, 2, \cdots, l)\, y_i < 0 \Rightarrow f(y_1, y_2, \cdots, y_l) < 0$$

(iv) $g_i, i = 1, 2, \cdots, l$ are continuous real-valued functions on $[t_0, \infty)$ such that

$$\lim_{t \to \infty} g_i(t) = \infty.$$

Throughout this paper, by "solution" of the differential equation (*) we shall mean only solutions x which are defined on a half-line $[t_x, \infty)$.

(*) Nella seduta dell'8 marzo 1975.

Also, we shall consider the oscillatory character of the solutions in the usual sense, i.e. a solution is called *oscillatory* if and only if it has no last zero, otherwise it is called *nonoscillatory*.

To obtain our result we make use of the following simple lemma, which has been proved very useful in the study of the oscillatory behavior of differential equations (Cfr. [9] and [10]).

LEMMA. *Consider the linear differential equation*

$$z' - \frac{\nu}{t} z + \frac{h(t)}{t} = 0 \tag{L}$$

where ν *is a positive integer and* h *is continuous real-valued function on the interval* $[T, \infty)$, $T > 0$.

If $\lim_{t\to\infty} |h(t)| = \infty$ *and* u *is a solution of* (L) *with* $u(T) = 0$, *then*

$$\lim_{t\to\infty} |u(t)| = \infty .$$

THEOREM. *Consider the differential equation* (*) *subject to the conditions* (i)-(iv) *and to the following one*:

(C) *For some integer* k, $0 \leqq k \leqq m - 1$ *and for every* $\mu_1 > 0$, $\mu_2 > 0$

$$\int^{\infty} t^k [\mu_1 a^+(t) - a^-(t) - \mu_2 b(t)] \, dt = \infty$$

or

$$\int^{\infty} t^k [\mu_1 a^-(t) - a^+(t) + \mu_2 b(t)] \, dt = \infty$$

where $a^+(t) = \max\{a(t), 0\}$ *and* $a^-(t) = \max\{-a(t), 0\}$.

Then every bounded solution x *of* (*) *is either oscillatory or such that*

$$\liminf_{t\to\infty} |x(t)| = 0 .$$

Proof. We assume that the conclusion of the theorem is not valid, then there exists a bounded nonoscillatory solution x of (*) with $\liminf_{t\to\infty} |x(t)| \neq 0$. Without loss of generality, we suppose that the domain of x is the whole interval $[t_0, \infty)$. Moreover, this solution is supposed positive, since the substitution $u = -x$ transforms (*) into an equation of the same form satisfying the assumptions of the theorem.

By (iii) and (iv), there exists a $T > \max\{t_0, 0\}$ and two positive constants c_1 and c_2 such that

$$c_1 \leqq f(x[g_1(t)], \cdots, x[g_l(t)]) \leqq c_2 \quad \text{for every } t \geqq T. \tag{1}$$

If

$$q_{ij}(t) = \int_T^t s^i [r(s)\, x^{(n-m)}(s)]^{(j)}\, ds\,, \qquad 1 \leqq i \leqq j$$

then integrating by parts, we obtain

$$q_{ij}(t) = t q'_{i-1,j-1}(t) - i q_{i-1,j-1}(t) - T^i [r(s)\, x^{(n-m)}(s)]_{s=T}^{(j-1)}.$$

Hence $q_{i-1,j-1}$ is a solution of the differential equation

$$(L_{ij}) \qquad z' - \frac{i}{t} z + \frac{h_{ij}(t)}{t} = 0$$

where $h_{ij}(t) = -q_{ij}(t) - T^i [r(s)\, x^{(n-m)}(s)]_{s=T}^{(j-1)}$. Obviously, this solution satisfies the initial condition $q_{i-1,j-1}(T) = 0$.

Now, for $i = k$ and $j = m$ we have

$$h_{km}(t) = -q_{km}(t) - T^k [r(s)\, x^{(n-m)}(s)]_{s=T}^{(m-1)} =$$

$$= \int_T^t s^k [a(s) f(x[g_1(s)], \cdots, x[g_l(s)]) - b(s)]\, ds - T^k [r(s)\, x^{(n-m)}(s)]_{s=T}^{(m-1)}$$

and consequently, by (1), we obtain

$$h_{km}(t) \geqq c_2 \int_T^t s^k \left[\frac{c_1}{c_2} a^+(s) - a^-(s) - \frac{1}{c_2} b(s)\right] ds - T^k [r(s)\, x^{(n-m)}(s)]_{s=T}^{(m-1)}$$

and

$$h_{km}(t) \leqq c_2 \int_T^t s^k \left[a^+(s) - \frac{c_1}{c_2} a^-(s) - \frac{1}{c_2} b(s)\right] ds - T^k [r(s)\, x^{(n-m)}(s)]_{s=T}^{(m-1)}$$

for every $t \geqq T$. Thus, by the assumptions of the theorem, we have that $\lim_{t\to\infty} | h_{km}(t) | = \infty$. Next, applying the lemma for the differential equation (L_{km}), we obtain

$$\lim_{t\to\infty} | q_{k-1,m-1}(t) | = \infty\,.$$

Furthermore, since

$$h_{k-1,m-1}(t) = -q_{k-1,m-1}(t) - T^{k-1} \,|\, r(s)\, x^{(n-m)}(s) \,|_{s=T}^{(m-2)}$$

we have

$$\lim_{t\to\infty} | h_{k-1,m-1}(t) | = \infty$$

and consequently, applying the lemma again for the differential equation $(L_{k-1,m-1})$, we obtain

$$\lim_{t\to\infty} | q_{k-2,m-2}(t) | = \infty\,.$$

Following the same procedure, we finally obtain

$$\lim_{t\to\infty} |q_{0,m-k}(t)| = \infty .$$

This, by virtue of

$$[r(t)\, x^{(n-m)}(t)]^{(n-k-1)} = [r(s)\, x^{(n-m)}(s)]_{s=T}^{(m-k-1)} + q_{0,m-k}(t)$$

gives

$$\lim_{t\to\infty} [r(t)\, x^{(n-m)}(t)]^{(m-k-1)} = \pm\infty$$

and consequently

$$\lim_{t\to\infty} r(t)\, x^{(n-m)}(t) = \pm\infty .$$

In the case where $\lim_{t\to\infty} r(t)\, x^{(n-m)}(t) = \infty$, there exists some $T_1 \geqq T$ such that

$$r(t)\, x^{(n-m)}(t) > 1 \qquad \text{or} \qquad x^{(n-m)}(t) > \frac{1}{r(t)}$$

for every $t \geqq T_1$. Thus

$$x^{(n-m-1)}(t) > x^{(n-m-1)}(T_1) + \int_{T_1}^{t} \frac{ds}{r(s)} \qquad \text{for every} \quad t \geqq T_1$$

which, by (ii), gives $\lim_{t\to\infty} x^{(n-m-1)}(t) = \infty$ and consequently $\lim_{t\to\infty} x(t) = \infty$, a contradiction.

In the other case, where $\lim_{t\to\infty} r(t)\, x^{(n-m)}(t) = -\infty$, following the same procedure we obtain the contradiction $\lim_{t\to\infty} x(t) = -\infty$.

Remark. In particular, for the differential equations

$$[r(t)\, x^{(n-1)}(t)]' + a(t) f(x[g(t)]) = b(t)$$

and

$$[r(t)\, x'(t)]^{(n-1)} + a(t) f(x[g(t)]) = b(t)$$

the above theorem leads to a generalization of a recent result of Kusano and Onose [6].

Also, the arguments used in this paper can be applied to extend the results obtained by the Authors in [9] and [10] to differential equations with deviating arguments of the more general form

$$[r(t)\, x^{(n-m)}(t)]^{(m)} + F(t, x\langle\sigma_0(t)\rangle, x'\langle\sigma_1(t)\rangle, \cdots, x^{(n-1)}\langle\sigma_{n-1}(t)\rangle) = 0$$

where

$$x^{(i)}\langle\sigma_i(t)\rangle = (x^{(i)}[\sigma_{i1}(t)], \cdots, x^{(i)}[\sigma_{i\lambda_i}(t)]), \qquad i = 1, 2, \cdots, n-1 .$$

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