Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Tadeusz Jankowski

On the convergence of approximate iterations for an abstract equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **58** (1975), n.3, p. 277–283.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_3_277_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica) SIMAI & UMI

http://www.bdim.eu/

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1975.

Analisi matematica. — *On the convergence of approximate iterations for an abstract equation.* Nota di TADEUSZ JANKOWSKI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questa Nota si studiano soluzioni approssimanti di una soluzione dell'equazione astratta $x = f(x)$.

1. INTRODUCTION

Many Authors have been interested in the solution of an abstract equation

$$x = f(x). \tag{1}$$

The most simple approximate solutions of this equation are the simple iterations $x_{n+1} = f(x_n)$, $n = 0, 1, \cdots$ (see [3]–[8]). Because of the rounding error or error in the evaluation of f, in place of the sequence $\{x_n\}$ another sequence $\{x_n^*\}$ is produced (for a more detailed discussion of this problem see [5]).

The sequence $\{x_n^*\}$ may be generated in a variety of ways. In papers [3]–[5], [7] it was considered the convergence problem for a sequence $\{x_n^*\}$ such that

$$\rho(x_{n+1}^*, f(x_n^*)) \searrow 0,$$

where the ρ has the properties of a metric. The approximate iterations x_n^* can be also of the form

$$x_{n+1}^* = f_n(x_n^*), \qquad n = 0, 1, \cdots,$$

(see [3]–[5], [7]), or

$$x_n^* = f_n(x_n^*), \qquad n = 0, 1, \cdots, \tag{2}$$

(see [1], [3]–[5], [8]), or of similar form (see [2]–[5]) where f_n is convergent to f.

In this paper we give conditions under which the sequence (2) and

$$g_n = h_n(g_n, g_{n-1}), \qquad n = 1, 2, \cdots,$$

are convergent to the solution of Eq. (1). There are estimations better than the known ones.

2. PRELIMINARIES AND LEMMAS

Let G denote a partially ordered set (an ordering relation is denoted by $<$; we write $u \leq v$ iff $u < v$ or $u = v$); there exists in G an element 0 such that $0 \leq u$ for any $u \in G$. Moreover, a relation $u + v$ is defined in G

(*) Nella seduta dell'8 marzo 1975.

and for any non-increasing sequence $\{u_n\} \subset G$ there exists a unique element $u \in G$ called the limit of the sequence $\{u_n\}$ and we write $u = \lim_{n \to \infty} u_n$ or $u_n \searrow u$ (the precise definition of the set G is in [6] or [4] or [2]).

Suppose that

Assumption H_1. The function $a : \Delta \to G$, $\Delta \subset G$ is non-decreasing and such that:

1) $0 \in \Delta$ and if $k \in \Delta$, then $u \in \Delta$ for any $u \leq k$;

2) if $u_n \in \Delta$, $n = 0, 1, \cdots$, and $u_n \searrow u$ then $a(u_n) \searrow a(u)$;

3) $u = 0$ is the only solution in Δ of the equation $u = a(u)$.

We state the following

LEMMA 1 (see [4]). *If Assumption* H_1 *is fulfilled, the sequence* $\{\delta_n\} \subset G$ *is convergent to the element* 0 *and there exists* $b \in \Delta$ *such that* $\delta_0 + a(b) \leq b$, *then for the sequence* $\{z_n\}$ *of the form*

$$z_0 = b \quad , \quad z_{n+1} = \delta_n + a(z_n) , \qquad n = 0, 1, \cdots,$$

we have $z_n \searrow 0$.

By induction we can prove the following

LEMMA 2. *Suppose that Assumption* H_1 *is fulfilled for the function* $d : \Delta \times \Delta \to G$, *and that the sequence* $\{c_n\} \subset G$ *is such that* $c_n \searrow 0$. *Moreover let the elements* u_0 *and* u_1 *of the sequence* $\{u_n\}$ *be such that* $u_1 \leq u_0$ *and* $c_1 + d(u_1, u_0) \leq u_1$ *and* $u_{n+1} = c_n + d(u_n, u_{n-1})$, $n = 1, 2, \cdots$.

Then we have $u_{n+1} \leq u_n$, $n = 0, 1, \cdots$, *and* $u_n \searrow 0$.

Proof. We have $u_2 = c_1 + d(u_1, u_0) \leq u_1 \leq u_0$. By induction we get $u_{n+1} \leq u_n$, $n = 0, 1, \cdots$. According to Assumption H_1 we obtain the assertion of the lemma.

Remark 1. It is easy to see that the assertion of Lemma 2 remains true if there exists $b \in \Delta$ such that $c_1 + d(b, b) \leq b$ and $u_1 = u_0 = b$.

3. THE MAIN SPACE R

Let R denote an abstract space such that for some sequences $\{x_n\} \subset R$ there exists a uniquely determined limit $\lim_{n \to \infty} x_n = x \in R$. Moreover, for any $x^* \in R$ and $b \in G$ the sphere $S(x^*, b) = [x \in R, \rho(x, x^*) \leq b]$ is a closed set where the function $\rho : R \times R \to G$ has the property of a metric. The space R is complete in the following sense: if $\{c_n\} \subset G$, $c_n \searrow 0$ and for $\{x_n\} \subset R$ the Cauchy condition $\rho(x_n, x_{n+m}) \leq c_n$, $n, m = 0, 1, \cdots$, is satisfied, then there exists a limit $y \in R$ of sequence $\{x_n\}$ (see also [4] and [6] or [2]).

Assumption H_2. The function $f : S(x^*, b) \to R$, $x^* \in R$, $b \in \Delta$ has the property:

1) for any $x, y \in S(x^*, b)$ we have

$$\rho(f(x), f(y)) \leq a(\rho(x, y)),$$

where the function a satisfies Assumption H_1 and $b + b \overset{\text{df}}{=} 2b \in \Delta$,

2) there exists a $q \in \Delta$ such that

$$\rho(x^*, f(x^*)) \leq q \qquad \text{and} \quad q + a(b) \leq b .$$

Assumption H_3. Suppose that

1) $f_n : S(x^*, b) \to R$, $n = 0, 1, \cdots$,

2) for any $x, y \in S(x^*, b)$

$$\rho(f_n(x), f_n(y)) \leq a_n(\rho(x, y)), \qquad n = 0, 1, \cdots,$$

where the functions a_n satisfy Assumption H_1 (except 3)), $2b \in \Delta$,

3) for any $v \in \Delta$ we have $a_n(v) \leq a(v)$, $n = 0, 1, \cdots$, where the function a satisfies Assumption H_1.

Assumption H_4. Suppose that

1) $h_n : S(x^*, b) \times S(x^*, b) \to R$, $n = 0, 1, \cdots$,

2) for any $x, y, s, t \in S(x^*, b)$, $n = 0, 1, \cdots$, we have

$$\rho(h_n(x, y), h_n(s, t)) \leq d_n(\rho(x, s), \rho(y, t)),$$

where the functions $d_n(d_n : \Delta \times \Delta \to G)$ satisfy Assumption H_1 (except 3)), $2b \in \Delta$,

3) there exists the function $d : \Delta \times \Delta \to G$ satisfying Assumption H_1 and such that for $u, v \in \Delta$ we have $d_n(u, v) \leq d(u, v)$, $n = 0, 1, \cdots$,

4) there exists a $q \in \Delta$ such that for $n = 0, 1, \cdots$, we have

$$\rho(x^*, h_n(x^*, x^*)) \leq q \qquad \text{and} \quad q + d(b, b) \leq b .$$

4. CONVERGENCE OF APPROXIMATE ITERATIONS

Let us recall the existence theorem from [6] (see also [4] and [2] or [3]).

THEOREM 1. *If Assumption* H_2 *is fulfilled then there exists in* $S(x^*, b)$ *a unique solution* $\bar{x}$ *of* Eq. (1) *which is the limit of the sequence* $\{x_n\}$ *of the form* $x_0 = x^*$, $x_{n+1} = f(x_n)$, $n = 0, 1, \cdots$.

We can now formulate two theorems on the convergence of sequences of approximate iterations to the solution $\bar{x}$ of Eq. (1). The sequences of approximate iterations may be defined by the relations $x_{n+1} = f_n(x_n)$ and $y_n = f_n(y_n)$ for $n = 0, 1, \cdots$, or $g_n = h_n(g_n, g_{n-1})$, $n = 1, 2, \cdots$, where $\{f_n\}$ and $\{h_n\}$ are sequences convergent to the function f (the elements y_n and g_n are fixed points of the function f_n or h_n respectively; see also 2) of Theorem 2 and 2) of Theorem 3). Now we get

THEOREM 2. *If Assumptions* H_2 *and* H_3 *are fulfilled and if*

1) $x_0 = x^*$, $x_{n+1} = f_n(x_n)$, $n = 0, 1, \cdots$,

2) $\rho(f_n(x_n), f(x_n)) \leq \delta_n \searrow \delta$, $\delta_0 \leq q$,

3) $\rho(f_n(x^*), x^*) \leq q$, $n = 0, 1, \cdots$,

4) *the sequence* $\{y_n\}$ *of the fixed points* y_n *of the functions* f_n *has the property:*

(a) $\rho(y_{n+1}, y_n) \leq \varepsilon_n \searrow \varepsilon, \varepsilon_0 \leq q$,

(b) $\rho(y_0, x^*) \leq b$,

then there exists in $S(x^*, b)$ *a unique solution* $\bar{x}$ *of Eq.* (1) *and*

$$\rho(x_n, \bar{x}) \leq z_n \quad , \quad \rho(y_n, \bar{x}) \leq z_n + \tilde{z}_n , \qquad n = 0, 1, \cdots, \tag{3}$$

where the sequence $\{\tilde{z}_n\}$ *is defined in Lemma* 1 *with* a_n *instead of* a *and with* ε_n *instead of* δ_n.

Moreover if $\varepsilon = \delta = 0$ *then*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \bar{x} .$$

Proof. We see that for any function y_n Assumption H_2 is satisfied: by Theorem 1 above, y_n exists and $y_n \in S(x^*, b)$, $n = 0, 1, \cdots$. Since $x_0 \in S(x^*, b)$ and

$$\begin{aligned} \rho(x_{n+1}, x^*) &\leq \rho(f_n(x_n), f_n(x^*)) + \rho(f_n(x^*), x^*) \\ &\leq a_n(\rho(x_n, x^*)) + q , \qquad n = 0, 1, \cdots, \end{aligned}$$

we get $x_n \in S(x^*, b)$, $n = 0, 1, \cdots$.

Further we have

$$\begin{aligned} \rho(x_{n+1}, y_{n+1}) &\leq \rho(f_n(x_n), f_n(y_n)) + \rho(y_n, y_{n+1}) \\ &\leq a_n(\rho(x_n, y_n)) + \varepsilon_n , \qquad n = 0, 1, \cdots. \end{aligned}$$

Put $w_n = \rho(x_n, y_n)$, $n = 0, 1, \cdots$; then we have

$$\begin{cases} w_0 \leq b , \\ w_{n+1} \leq a_n(w_n) + \varepsilon_n , \qquad n = 0, 1, \cdots. \end{cases}$$

By induction, we obtain $w_n \leq \tilde{z}_n \leq z_n$, $n = 0, 1, \cdots$, and from Lemma 1 with $\varepsilon = \delta = 0$ we get the assertion of Theorem 2.

Remark 2. Theorem 2 is similar to Theorems 2 and 3 [4]. In paper [4] the condition 4) is replaced by $\rho(y_n, \bar{x}) \leq b$, $n = 0, 1, \cdots$, and there are given the adequate estimations with the function a instead of a_n.

Remark 3. Let

$$a(u) = Lu \quad , \quad L \in [0, 1) \quad , \quad u \in \Delta.$$

Then Assumption H_1 and the condition 3) of Assumption H_2 are satisfied $b \geq q(1 - L)^{-1}$. Moreover if we put

$$a_n(u) = L_n u , L_n \in [0, L] , u \in \Delta , \qquad n = 0, 1, \cdots,$$

then the estimations (3) are fulfilled with the sequences $\{z_n\}$ and $\{\tilde{z}_n\}$ of the form

$$\begin{cases} z_0 = b\,, \\ z_{n+1} = \sum_{i=0}^{n} \delta_{n-i}\, L^i + L^{n+1}\, b\,, \end{cases} \qquad n = 0, 1, \cdots,$$

and

$$\tilde{z}_0 = b\,,$$

$$\tilde{z}_{n+1} = \sum_{j=0}^{n} \prod_{i=j}^{n} L_i\, \varepsilon_{j-1} + \varepsilon_n\,, \ \varepsilon_{-1} \overset{\text{df}}{=} b\,, \qquad n = 0, 1, \cdots.$$

Theorem 2 contains some results of [1] for the case $L_n = L$, $n = 0, 1, \cdots$.

Remark 4. If the condition (*a*) of Theorem 2 replaced by: there exists $\bar{y} \in S(x^*, b)$ such that

$$\rho(y_n, \bar{y}) \leq \delta_n \searrow \delta \quad , \quad \delta_0 + \delta_1 \leq q,$$

then

$$\rho(x_n, \bar{y}) \leq p_n\,, \qquad n = 0, 1, \cdots,$$

where $p_0 = b$, $p_{n+1} = a_n(p_n) + 2\,\delta_n + \delta_{n+1}$, $n = 0, 1, \cdots$.

Moreover, if $\delta = 0$ then

$$\lim_{n\to\infty} x_n = \bar{y}\,.$$

Indeed, it is easy to see that

$$w_0 \leq b\,,$$

$$w_{n+1} \leq a_n(w_n) + \delta_n + \delta_{n+1}\,, \qquad n = 0, 1, \cdots,$$

where $w_n = \rho(x_n, y_n)$ (see the proof of Theorem 2).

Now, if $\delta = 0$ then we have

$$\rho(x_n, \bar{y}) \leq \rho(x_n, y_n) + \rho(y_n, \bar{y}) \searrow 0\,.$$

THEOREM 3. *If Assumption* H_4 *is fulfilled and if*

1) $f : S(x^*, b) \to R$.

2) $\rho(h_n(x, x), f(x)) \leq \varepsilon_n \searrow 0$, $x \in S(x^*, b)$,

3) $\rho(f_n(x_n), h_n(x_n, x_{n-1})) \leq p_n \searrow 0$, $p_1 + \varepsilon_1 \leq q$,

then there exists in $S(x^*, b)$ *the unique solution* $\bar{x}$ *of Eq.* (1) *and the sequence* $\{g_n\}$ *is well-defined by the relations*

$$g_0 = x^* \quad , \quad g_n = h_n(g_n, g_{n-1}), \qquad n = 0, 1, \cdots.$$

Further if

4) $\rho(g_n, g_{n+1}) \le \delta_n \searrow \delta$, $\delta_0 + \varepsilon_0 \le q$,

5) $x_0 = \bar{x}$, $x_{n+1} = f_n(x_n)$, $n = 0, 1, \cdots$, *and* $\rho(x_1, \bar{x}) \le b$, $\rho(x_1, g_1) \le b$,

then

$$\rho(x_n, \bar{x}) \le \tilde{u}_n + k_n \quad \text{and} \quad \rho(g_n, \bar{x}) \le \tilde{u}_n, \qquad n = 0, 1, \cdots, \tag{4}$$

where

$$\begin{cases} \tilde{u}_0 = \tilde{u}_1 = b, \\ \tilde{u}_{n+1} = \varepsilon_n + \delta_n + d_n(\tilde{u}_n, \tilde{u}_{n-1}), \end{cases} \qquad n = 1, 2, \cdots,$$

and

$$\begin{cases} k_0 = k_1 = b, \\ k_{n+1} = p_n + \delta_n + d_n(k_n, k_{n-1}), \end{cases} \qquad n = 1, 2, \cdots.$$

Moreover if $\delta = 0$ *then*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} g_n = \bar{x}. \tag{5}$$

Proof. In paper [4] it is proved that Assumption H_2 with $a(u) = d(u, u)$ holds true, and consequently there exists the unique solution $\bar{x}$ of Eq. (1).

Now we see that

$$\begin{aligned} \rho(x_{n+1}, g_{n+1}) \le \rho(f_n(x_n), h_n(x_n, x_{n-1})) + \\ + \rho(h_n(x_n, x_{n-1}), h_n(g_n, g_{n-1})) + \rho(g_n, g_{n+1}) \le \\ \le p_n + \delta_n + d_n(\rho(x_n, g_n), \rho(x_{n-1}, g_{n-1})), \qquad n = 1, 2, \cdots. \end{aligned}$$

Further, by induction we can prove

$$\rho(x_n, g_n) \le k_n, \qquad n = 0, 1, \cdots,$$

and by Lemma 2 we get $k_n \searrow 0$.

Finally, we have

$$\begin{aligned} \rho(g_{n+1}, \bar{x}) \le \rho(g_{n+1}, g_n) + \rho(h_n(g_n, g_{n-1}), h_n(\bar{x}, \bar{x})) + \rho(h_n(\bar{x}, \bar{x}), f(\bar{x})) \le \\ \le \delta_n + \varepsilon_n + d_n(\rho(g_n, \bar{x}), \rho(g_{n-1}, \bar{x})), \qquad n = 1, 2, \cdots, \end{aligned}$$

and hence we obtain the estimations (4).

Now if $\delta = 0$ we get our assertion (5).

REFERENCES

[1] R.R. KLUGE und G. BRUCKNER (1972) – *Iterationsverfahren bei Folgen von Abbildungen und nichtlineare Probleme mit monotonen Operatoren*, «Math. Nachr.», *54*, 335–353.

[2] T. JANKOWSKI (1971) – *Convergence of generalized-approximate iterations for an abstract equation*, «Ann. Polon. Math.», *24*, 269–280.

[3] N. S. KURPEL (1968) – *Projection–iterative methods of solution of operator equations* (in Russian), Kiev.

[4] M. KWAPISZ (1969) – *On the convergence of approximate iterations for an abstract equation*, «Ann. Polon. Math.», *22*, 73–87.

[5] J. M. ORTEGA and W. C. RHEINBOLDT (1967) – *On a class of approximate iterative pro*cesses, «Arch. Rat. Mech. Anal.», *23*, 352–365.

[6] T. WAŻEWSKI (1960) – *Sur un procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison*, «Bull. Acad. Sci., sér. Sci. math. astr. et phys.», *8*, 45–52.

[7] A. PASQUALI (1969) – *Alcune proprietà di una classe di procedimenti iterativi approssimati connessi con trasformazioni più generali delle contrazioni, in uno spazio metrico*, «Annali di Mat. pura e appl.», *82*, 35–48.

[8] A. PASQUALI (1969) – *Su una classe di procedimenti iterativi per la risoluzione di equazioni funzionali e applicazioni*, «Annali di Mat. pura e appl.», *83*, 213–225.