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Some reduction formulas for the Poincaré series of modules

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Algebra. — *Some reduction formulas for the Poincaré series of modules*. Nota di FRANCO GHIONE e TOR H. GULLIKSEN, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Sia $(R, \mathfrak{m})$ un anello locale, e M un modulo di tipo finito su di esso. Si danno alcuni risultati sulla razionalità della serie di Poincaré di $M : P_M^R(t) = \sum_{p=0}^{\infty} \dim_k \operatorname{Tor}_p^R(k, M)\, t^p$, $k = R/\mathfrak{m}$ provando in casi particolari la verità di una congettura di Serre–Kaplanski.

INTRODUCTION

In this paper we shall investigate the Poincaré series of a finitely generated module M over a local (noetherian) ring $R, \mathfrak{m}$, that is the power series

$$P_M^R(t) := \Sigma_{p \geq 0} \dim_k \operatorname{Tor}_p^R(k, M)\, t^p$$

k being the residue field of R. $P_k^R(t)$ will be called the Poincaré series of the ring R. Although not much is known about $P_M^R(t)$ in general, there is evidence to believe that this power series always represents a rational function. This is of course the case if R is a regular local ring, in which case $P_M^R(t)$ is a polynomial of degree not exceeding the global dimension of R. Recently it has been shown that $P_M^R(t)$ is a rational function if R is a local complete intersection, Gulliksen [3].

In the present paper we shall establish the rationality of $P_M^R(t)$ also in the case where R is a Golod ring (see Section 2 for the definition). It is known that if R is a factor of a regular ring by two relations, or if R has imbedding dimension less than or equal to two, then R is either a complete intersection or a Golod ring. Hence the rationality of $P_M^R(t)$ is established in those cases. This generalizes results of Shamash [5] and Scheja [6] who worked with the case $M = k$.

In Theorem 5 in Section 3 we give the following reduction formula:

Let y be a regular element in $\mathfrak{m}$. Let $\mathfrak{a}$ be an ideal in R and put $R' := R/y\mathfrak{a}$. If $\mathfrak{a}M = 0$ then

$$(i) \qquad P_M^{R'}(t) = P_M^R(t)\,[1 - t(\alpha(t) - 1)]^{-1}$$

where $\alpha(t) = P_{R/\mathfrak{a}}^R(t)$.

In Section 4 we give applications of Theorem 5 and the results of Section 3. Several examples are worked out. In particular it is shown that

(*) Nella seduta dell'8 febbraio 1975.

if k is a field and $\mathfrak{a}$ is an ideal generated by monomials in the ring $A = k[[X_1, X_2, X_3]]$, then the Poincaré series of $A/\mathfrak{a}$ is rational.

In the last section we remark that in order to prove the rationality of $P_M^R(t)$ for all R and M, it suffices to prove the rationality of $P_{R/\mathfrak{m}}^R(t)$ for all local rings R of dimension zero.

NOTATIONS

If $N = \Pi_{p \geq 0} N_p$ is a graded R-module where each homogeneous component N_p is a free R-module of finite rank, we let $\chi_R(N)$ or simply $\chi(N)$ denote the power series

$$\Sigma_{p \leq 0} \operatorname{rank}(N_p)\, t^p .$$

The term " R-algebra " will be used in the sense of Tate [7]. By an augmented R-algebra F we will mean an R-algebra F with a surjective augmentation map $F \to R/\mathfrak{m}$ which is a homomorphism of R-algebras. Recall that the Koszul complex generated over R by a minimal set of generators for $\mathfrak{m}$ is an R-algebra which up to (a non-canonical) isomorphism depends only on the ring R. Thus we shall talk about the Koszul complex of R.

1. ON MASSEY OPERATIONS

Let F be an augmented R-algebra with a trivial Massey operation γ, and let S be the set of cycles associated with γ. For the definitions and details the reader is referred to Gulliksen [2]. Recall that S represents a minimal set of generators for the kernel of the map $H(F) \to R/\mathfrak{m}$ induced by the augmentation on F. γ is a function with values in F, defined on the set of finite sequences of elements in S such that $\gamma(z) = z$ for $z \in S$. By means of F and γ it is possible to construct an R-free resolution of $R/\mathfrak{m}$. We will briefly recall the construction.

To each cycle z in S select a symbol u of degree one more than the degree of z. Let $N = \Pi_q N_q$ be the free graded R-module generated by the set of selected symbols u. Let $T = T_R(N)$ be the tensor algebra generated over R by N. Put

$$X := F \otimes_R T .$$

By means of the canonical map $F \to F \otimes T$, sending f to $f \otimes 1$, F will be considered as a submodule of X. We will now extend the differential d on F to a differential on X (also denoted by d) in the following way: It suffices to define d on a set of generators for the R-module X. If f is a homogeneous element in F of degree $\deg f$ and if $u_1, \cdots, u_n$ ($n \geq 1$) are selected symbols corresponding to the cycles $z_1, \cdots, z_n$ in S we put

$$d(f \otimes u_1 \otimes \cdots \otimes u_n) = d(f \otimes u_1 \otimes \cdots \otimes u_{n-1}) \otimes u_n + (-1)^{\deg f} f\gamma(z_1, \cdots, z_n).$$

One can show that $d^2 = 0$ and that X is an R-free resolution of k. Cf. [2]. Moreover, if F is minimal in the sense that $dF \subseteq \mathfrak{m}F$, and if $\operatorname{Im} \gamma \subseteq \mathfrak{m}F$, then X is a minimal resolution.

DEFINITION. The resolution X constructed above will be called the Golod extension of the couple (F, γ) and will be denoted $X = (F, \gamma, N)$.

THEOREM 1. *Let* R *be a local ring with residue field* k *and let* $\mathfrak{a}$ *be an ideal in* R. *Let* M *be a finitely generated* R*-module of finite projective dimension such that* $\mathfrak{a}M = 0$. *Let* F *be an augmented* R*-algebra which is an* R*-free resolution of* k. *Put* $R' := R/\mathfrak{a}$, $F' := F/\mathfrak{a}F$ *and assume that* F′ *has a trivial Massey operation* γ. *Then there exists a polynomial* $\pi(t)$ *with integral coefficients such that*

$$P_M^{R'}(t) = \pi(t)\,[1 - t(P_{R'}^{R}(t) - 1)]^{-1}.$$

Proof. Let $X = (F', \gamma, N)$ be the Golod extension of (F', γ). Then we have an identity of graded R-modules

$$X = F' \oplus X \otimes N. \tag{1}$$

We let Y be the complex whose underlying graded module is $X \otimes N$, and whose differential is $d \otimes 1_N$, d being the differential on X. (1) leads to an exact sequence of complexes

$$0 \longrightarrow F' \xrightarrow{\alpha} X \xrightarrow{\beta} Y \longrightarrow 0 \tag{2}$$

where α is the canonical injection, and β is the canonical projection onto the second factor. Since $\mathfrak{a}M = 0$ and M has finite projective dimension we have

$$H_p(M \otimes_{R'} F') = H_p(M \otimes_R F) = \operatorname{Tor}_p^R(M, k) = 0$$

for all p sufficiently large. Hence from (2) we obtain

$$H_p(M \otimes X) \cong H_p(M \otimes Y) \tag{3}$$

for all p sufficiently large. Hence we have

$$\operatorname{Tor}_p^{R'}(M, k) = H_p(M \otimes X) \cong H_p(M \otimes Y) = \Pi_q H_{p-q}(M \otimes X) \otimes N_q$$

for all p sufficiently large. Thus there exists a polynomial $\pi(t)$ with integral coefficients such that

$$P_M^{R'}(t) = \chi(H(M \otimes X) \otimes N) + \pi(t) = P_M^{R'}(t)\,\chi(N) + \pi(t).$$

This yields the desired formula since

$$\chi(\mathrm{N}) = t(\chi(\mathrm{H}(\mathrm{F}')) - 1) = t(\mathrm{P}^{\mathrm{R}}_{\mathrm{R}'}(t) - 1).$$

2. MODULES OVER GOLOD RINGS

We will first recall the definition of a Golod ring. Let R be a local ring with maximal ideal $\mathfrak{m}$, and let K be the Koszul complex of R. R is called a Golod ring if the canonically augmented R-algebra K has a trivial Massey operation in the sense of Gulliksen [2]. This is equivalent to saying that all Massey operations on H (K) vanish in the sense of Golod [1]. The following result shows that Golod rings can be characterized entirely in terms of the Poincaré series. The proposition is due to Golod, and the proof can be found in [1].

PROPOSITION 2. *A local ring* R, $\mathfrak{m}$ *is a Golod ring if and only if*

$$\mathrm{P}^{\mathrm{R}}_{\mathrm{R}/\mathfrak{m}}(t) = (1+t)^n [1 - c_1 t^2 - c_2 t^3 - \cdots - c_n t^{n+1}]^{-1}$$

where $n = \dim \mathfrak{m}/\mathfrak{m}^2$ *and* $c_i = \dim \mathrm{H}_i(\mathrm{K})$ *for* $1 \leq i \leq n$.

Examples of Golod rings:

I) The rings of the form $k[[\mathrm{X}_1, \mathrm{X}_2]]/\mathfrak{a}$ (where k is a field), which are not complete intersections. That these rings are Golod rings follows from Satz 9 in Scheja [6] and Proposition 2 above.

II) The rings $\mathrm{A}/y\mathfrak{a}$ where A is a regular local ring and y is a non-unit in A. Cf. Schamash [5].

III) $\mathrm{A}/(a_1, a_2)$ where A is regular, and a_1 and a_2 do not form a regular sequence. This is a special case of Example II.

IV) $k[\mathrm{X}_1, \cdots, \mathrm{X}_n]/(\mathrm{X}_1, \cdots, \mathrm{X}_n)^r$. ($k$ is a field). Cf. Golod [1].

PROPOSITION 3. *Let* R, $\mathfrak{m}$ *be a local ring and let* $y \in \mathfrak{m} - \mathfrak{m}^2$ *be a regular element. Then* R *is a Golod ring if and only if* $\mathrm{R}/y\mathrm{R}$ *is a Golod ring.*

Proof. Let K be the Koszul complex of R. Put $k = \mathrm{R}/\mathfrak{m}$. Since the k-algebra H (K) and the Poincaré series $\mathrm{P}^{\mathrm{R}}_{\mathrm{R}/\mathfrak{m}}(t)$ are invariant under $\mathfrak{m}$-adic completion, Proposition 2 shows that there is no loss of generality in assuming that R is complete. Hence by the Cohen structure theorem we may assume that $\mathrm{R} = \mathrm{A}/\mathfrak{a}$ where A is a regular ring and $\mathfrak{a}$ is an ideal contained in the square of the maximal ideal of A. Let y' be an element of A that represents y in R. We have an isomorphism of k-vector spaces.

$$\mathrm{H}_i(\mathrm{K}) \cong \mathrm{Tor}^{\mathrm{A}}_i(\mathrm{R}, k). \tag{1}$$

Similarly the homology of the Koszul complex of R/yR is isomorphic to $\mathrm{Tor}^{A/y'A}(R/yR, k)$. Since y' is regular on A and R, and since $y'k = 0$ we have a canonical isomorphism

$$\mathrm{Tor}^{A/y'A}(R/yR, k) \cong \mathrm{Tor}^{A}(R, k) \tag{2}$$

It follows from Satz 1 in Scheja [6] that

$$P_k^R(t) = (1 + t)\, P_k^{R/yR}(t)\,. \tag{3}$$

Let $\bar{\mathfrak{m}}$ denote the maximal in R/yR. We have

$$\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 + 1\,. \tag{4}$$

The proposition now follows from Proposition 2 using (1), (2), (3) and (4).

THEOREM 4. *Let* $R, \mathfrak{m}$ *be a local Golod ring and let* M *be a finitely generated* R-*module. Then* $P_M^R(t)$ *is a rational function.*

Proof. Let $\hat{R}$ and $\hat{M}$ be the $\mathfrak{m}$-adic completions of R and M. It can easily be shown that $P_M^R(t) = P_{\hat{M}}^{\hat{R}}(t)$, hence as in the Proof of Proposition 3 we may assume that $R = A/\mathfrak{a}$ where A is a regular local ring and $\mathfrak{a}$ is an ideal which is contained in the square of the maximal ideal of A. Let K be the Koszul complex of A. Then $K' := K/\mathfrak{a}K$ is the Koszul complex of R. Since R is a Golod ring, K' has a trivial Massey operation. Since M has finite projective dimension over the regular ring A, Theorem 2 gives that $P_M^R(t)$ is a rational function.

3. Reduction formulas for the Poincaré series

THEOREM 5. *Let* $R, \mathfrak{m}$ *be a local ring, let* y *be a regular element in* $\mathfrak{m}$, *and let* $\mathfrak{a}$ *be an ideal in* R. *Put* $R' = R/y\mathfrak{a}$ *and* $\alpha(t) = P_{R/\mathfrak{a}}^R(t)$. *Then*

(i) *If* M *is an* R-*module such that* $\mathfrak{a}M = 0$, *then*

$$P_M^{R'}(t) = P_M^R(t)\,[1 - t(\alpha(t) - 1)]^{-1}\,.$$

(ii) *If* $\mathfrak{a} \neq R$ *and if* M *is an* R-*module of finite projective dimension such that* $y\mathfrak{a}M = 0$, *then*

$$P_M^{R'}(t) = \pi(t)\,[1 - t(\alpha(t) - 1)]^{-1}$$

where $\pi(t)$ *is a polynomial with integral coefficients.*

Proof. Let $F \to k$ be an augmented R-algebra which is an R-free resolution of $k = R/\mathfrak{m}$. Put $F' := F/y\mathfrak{a}F$. We are going to construct a trivial Massey operation on F'. First we remark that there exists an element a' of degree one in F' such that every cycle z' in F' is homologue with a cycle of the form $a'x'$ where x' is in $\mathfrak{a}F'$. In fact let $z \in F$ represent a given cycle z' in F'. Then we may write $dz = yx$ where $x \in \mathfrak{a}F$. Since

$$0 = d^2 z = y dx$$

and since y is regular in R we have $dx = 0$. Let $a \in F_1$ be such that $da = y$. We have

$$d(z - ax) = 0 .$$

Since F is acyclic, $z - ax$ is a boundary. Hence $z' - a'x'$ is a boundary in F', where $a'x'$ is the image of ax in F'.

Now choose a basis for the k-vector space $\operatorname{Ker}(H(F') \to k)$. Let S be a set of cycles in F' representing that basis, and choose S such that the cycles z' in S have the form $z' = a'x'$ where $x' \in \mathfrak{a}F'$. Since a' has degree 1, we have $(a')^2 = 0$. Thus we can construct a trivial Massey operation γ on F' by putting

$$\gamma(z') = z' \qquad \text{for} \quad z' \in S$$

$$\gamma(z'_1, \cdots, z'_n) = 0 \qquad \text{for} \quad n \geq 2 \quad ; \quad z'_1, \cdots, z'_n \in S .$$

Now (ii) follows from Theorem 2 since we have

$$(1) \qquad P^R_{R'}(t) = P^R_{R/\mathfrak{a}}(t) = \alpha(t) \qquad \text{for} \quad \mathfrak{a} \neq R .$$

We are now going to prove (i). Let $X = (F', \gamma, N)$ be the Golod extension of the couple (F', γ). Assume that $\mathfrak{a}M = 0$. Since $\operatorname{Im} \gamma \subseteq \mathfrak{a}F'$ the following diagram is commutative

$$\begin{array}{ccc} M \otimes X & \longrightarrow & M \oplus F' \oplus (M \otimes X) \oplus N \\ \big\downarrow 1_M \otimes d & & \big\downarrow 1_M \otimes d \oplus 1_M \otimes d \otimes 1_N \\ M \otimes X & \longrightarrow & M \otimes F' \oplus (M \otimes X) \otimes N \end{array}$$

Here the horizontal isomorphisms are induced by the identity (1) in the Proof of Theorem 2. The diagram yields

$$(2) \qquad H(M \otimes X) \cong H(M \otimes F') \oplus H(M \otimes X) \otimes N .$$

We have $M \otimes F' \cong M \otimes F$. Hence if $\mathfrak{a} \neq R$ the desired formula in (i) follows from (1) and (2). If $\mathfrak{a} = R$, then $M = 0$ and in this case the formula in (i) is trivial.

COROLLARY 6. *Let $\mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_r$ be a chain of ideals in a local ring $R, \mathfrak{m}$ $(r \geq 1)$. Let $y_1, \cdots, y_r$ be a sequence of elements in $\mathfrak{m}$ such that y_1 is regular in R, and for every i $(1 \leq i \leq r - 1)$ y_{i+1} is regular on*

$$R^i := R / \sum_{h=1}^{i} y_h \mathfrak{a}_h .$$

Let M *be an* R*-module such that* $\mathfrak{a}_r \mathrm{M} = 0$. *Then we have*

$$P_{\mathrm{M}}^{\mathrm{R}^r}(t) = P_{\mathrm{M}}^{\mathrm{R}}(t)\,[(1+t)^r - t \sum_{0 \leq p < r} (1+t)^p \alpha_{r-p}(t)]^{-1}$$

where $\alpha_q(t) = P_{\mathrm{R}/\mathfrak{a}_q}^{\mathrm{R}}(t)$ *for* $1 \leq q \leq r$.

In particular, if R *is a local complete intersection or a Golod ring, then* $P_{\mathrm{M}}^{\mathrm{R}^r}(t)$ *represents a rational function.*

Proof. The formula will be proved by induction on r. For $r = 1$ the formula is valid by Theorem 5. Now let i be an integer such that $1 \leq i < r$. Put

$$\beta_i(t) = (1+t)^i - t \sum_{p=0}^{i-1} (1+t)^p \alpha_{i-p}(t).$$

By induction we may assume that we have

$$P_{\mathrm{Q}}^{\mathrm{R}^i}(t) = P_{\mathrm{Q}}^{\mathrm{R}}(t)\,\beta_i(t)^{-1} \tag{1}$$

for all R-modules Q such that $\mathfrak{a}_i \mathrm{Q} = 0$. Now let L be an R-module such that $\mathfrak{a}_{i+1} \mathrm{L} = 0$. We are going to show that (1) remains valid if i is replaced by $i + 1$, and Q is replaced by L. Since $\mathfrak{a}_i \subseteq \mathfrak{a}_{i+1}$, we have $\mathfrak{a}_i(\mathrm{R}/\mathfrak{a}_{i+1}) = 0 = \mathfrak{a}_i \mathrm{L}$, so (1) yields

$$P_{\mathrm{R}/\mathfrak{a}_{i+1}}^{\mathrm{R}^i}(t) = \alpha_{i+1}(t)\,\beta_i(t)^{-1} \tag{2}$$

and

$$P_{\mathrm{L}}^{\mathrm{R}^i}(t) = P_{\mathrm{L}}^{\mathrm{R}}(t)\,\beta_i(t)^{-1}. \tag{3}$$

Since y_{i+1} is regular in R^i, Theorem 5 gives

$$P_{\mathrm{L}}^{\mathrm{R}^{i+1}}(t) = P_{\mathrm{L}}^{\mathrm{R}^i}(t)\,[1 - t\,(P_{\mathrm{R}/\mathfrak{a}_{i+1}}^{\mathrm{R}^i}(t) - 1)]^{-1}. \tag{4}$$

Substituting (2) and (3) in (4) and using the identity

$$\beta_{i+1}(t) = (1+t)\,\beta_i(t) - t\alpha_{i+1}(t)$$

we obtain the desired result.

We shall now give a lemma which gives conditions implying the hypothesis in the previous corollary. With the notation of that corollary we have

LEMMA 7. *Let* $\mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_r$ *be a sequence of ideals in* R. *Let* $y_1, \cdots, y_r$ *be a regular sequence contained in the maximal ideal and assume that* $y_1, \cdots, y_{i+1}$ *is a regular sequence for* $\mathrm{R}/\mathfrak{a}_i$ *for all* i $(1 \leq i \leq r - 1)$. *Then* y_{i+1} *is* R^i*-regular for all* i.

Proof. We will prove the proposition by induction on r, the number of ideals. For $r = 1$ there is nothing to prove. Now let $r \geq 2$ and

$1 \leq i \leq r-1$. Let λ be an element of R such that

$$\lambda y_{i+1} \in \sum_{h=1}^{i} y_h \mathfrak{a}_h . \tag{1}$$

It suffices to show that $\lambda \in \sum_{h=1}^{i} y_h \mathfrak{a}_h$. Reading (1) modulo y_i R, and using the induction hypothesis one obtains

$$\lambda \in \sum_{h=1}^{i-1} y_h \mathfrak{a}_h + y_i \mathrm{R} .$$

Hence we may write

$$\lambda = \sum_{h=1}^{i-1} y_h a_h + y_i a \tag{2}$$

where $a_h \in \mathfrak{a}_h$ and $a \in \mathrm{R}$. From (1) we obtain $\lambda y_{i+1} \in \mathfrak{a}_i$. Hence we have $\lambda \in \mathfrak{a}_i$. Now (2) yields

$$y_i a \in \mathfrak{a}_i$$

hence $a \in \mathfrak{a}_i$, so $\lambda \in \sum_{h=1}^{i} y_h \mathfrak{a}_h$.

We will end this section by giving an example where Lemma 7 can be applied.

Let S be a local ring, let $r \geq 1$ be an integer and let $\mathfrak{a}_1' \subseteq \cdots \subseteq \mathfrak{a}_r'$ be ideals in S. Put $\mathrm{R} := \mathrm{S}[[y_1, \cdots, y_r]]$ and put $\mathfrak{a}_i = \mathfrak{a}_i' \mathrm{R}$ for $1 \leq i \leq r$. Then the sequences $y_1, \cdots, y_r$ and $\mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_r$ satisfy the hypothesis in Corollary 6.

4. EXAMPLES

I) Let R be a local ring and let $\mathfrak{a}$ be an ideal in R. Let M be an R-module such that $\mathfrak{a}\mathrm{M} = 0$. Let $y_1, \cdots, y_r$ be a regular sequence in R which is contained in $\mathfrak{m}$, and assume that $y_1, \cdots, y_r$ is also regular sequence on $\mathrm{R}/\mathfrak{a}$. Put $\mathrm{R}' := \mathrm{R}/(y_1, \cdots, y_r)\,\mathfrak{a}$. Then Lemma 7 and Corollary 6 yields the formula

$$\mathrm{P}_{\mathrm{M}}^{\mathrm{R}'}(t) = \mathrm{P}_{\mathrm{M}}^{\mathrm{R}}(t)\,[\alpha(t) - \alpha(t)\,\beta(t) + \beta(t)]^{-1}$$

where $\alpha(t) = \mathrm{P}_{\mathrm{R}/\mathfrak{a}}^{\mathrm{R}}(t)$ and $\beta(t) = \mathrm{P}_{\mathrm{R}/(y_1, \cdots, y_r)}^{\mathrm{R}}(t) = (1+t)^r$.

In particular if R is a local complete intersection or a Golod ring, then $\mathrm{R}_{\mathrm{M}}^{\mathrm{R}'}(t)$ represents a rational function.

II) Let A be a regular local ring of dimension n. Let r and s be integers such that $0 \leq s \leq r \leq n$. Let $y_1, \cdots, y_r$ be a regular sequence in $\mathfrak{m}^2$ and let u be an element in $\mathfrak{m}$ such that $y_1, \cdots, y_s, u$ is a regular sequence. Put

$$\mathfrak{a} = (y_1, \cdots, y_s \;\;,\;\; u y_{s+1}, \cdots, u y_r) .$$

Then considering $A/\mathfrak{a}$ as a factor ring of the complete intersection $A/(y_1, \cdots, y_s)$, one easily deduces the following formula from Theorem 5:

$$P_k^{A/\mathfrak{a}}(t) = (1+t)^{n-s-1}[(1-t)_s(1-t(1+t)^{r-s-1})]^{-1}.$$

III) Let k be a field and consider the following ring of formal power series $A = k[[X_1, \cdots, X_n, Y_1, \cdots, Y_r]]$. Let $\mathfrak{a}'_1 \subseteq \cdots \subseteq \mathfrak{a}'_r$ be a chain of ideals in $A' = k[[X_1, \cdots, X]]_n$.

Put $\mathfrak{a}_i = \mathfrak{a}'_i A$ for $1 \leq i \leq r$, and put

$$\mathfrak{a} = \sum_{i-1}^{r} Y_i \mathfrak{a}_i .$$

Then for each A-module M such that $\mathfrak{a}_r M = 0$ we have

$$P_M^{A/\mathfrak{a}}(t) = P_M^A(t)\left[(1+t)^r\left(1 - t\sum_{0\leq p<r}(1+t)^p \alpha'_{r-p}(t)\right)\right]^{-1}$$

where $\alpha'_q(t) = P_{A'/\mathfrak{a}'_q}^{A'}(t)$. Clearly $P_M^{A/\mathfrak{a}}(t)$ is a rational function. In particular, if k' denotes the residue field of $A/\mathfrak{a}$ we get

$$P_{k'}^{R/\mathfrak{a}}(t) = (1+t)^n\left[1 - t\sum_{0\leq p<r}(1+t)^p \alpha'_{r-p}(t)\right]^{-1}.$$

IV) Let $\mathfrak{a}$ be an ideal generated by monomials in the ring $A = k[[X_1, X_2, X_3]]$, where k is a field. We shall also let k denote the residue field of $A/\mathfrak{a}$. We will show that $P_k^{A/\mathfrak{a}}(t)$ is rational.

We may write

$$\mathfrak{a} = \mathfrak{a}_1 X_1 + \mathfrak{a}_2$$

where $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are ideals in A and $k[[X_2, X_3]]$ respectively. Put $R = k[[X_2, X_3]]/\mathfrak{a}_2$. Then R is either a complete intersection or a Golod ring. See Example I in Section 2. By Proposition 3 we see that the same holds for $R[[X_1]]$. Hence $P_M^{R[[X_1]]}(t)$ is a rational function for every $R[[X_1]]$-module M. Since

$$A/\mathfrak{a} \cong (k[[X_2, X_3]]/\mathfrak{a}_2)[[X_1]]/\mathfrak{a}_1 X_1 = R[[X_1]]/\mathfrak{a}_1 X_1$$

it follows from Theorem 5 that $P_M^{A/\mathfrak{a}}(t)$ is rational for every module M such that $\mathfrak{a}_1 M = 0$. In particular $P_k^{A/\mathfrak{a}}(t)$ is rational.

Using Theorem 5 it is also possible to prove that the ring $k[[X_1, \cdots, X_n]]/(m_1, m_2, m_3)$ has rational Poincaré series, m_1, m_2, m_3 being monomials.

5. Reduction to the case of dimension zero

Proposition 8. *The following statements are equivalent*:

(i) $P_{R/\mathfrak{m}}^R(t)$ *is rational for every local ring* $R, \mathfrak{m}$ *of dimension zero.*

(ii) $P_M^R(t)$ *is rational for every local ring* R *and every finitely generated* R-*module* M.

Proof. It suffices to prove (i) $\Rightarrow$ (ii). Suppose that $P_k^R(t)$ is rational for every local ring of dimension zero. From Theorem 3.17 in Levin [4] one deduces that $P_{R/\mathfrak{m}}^R$ is rational for every local ring R, $\mathfrak{m}$. By Theorem 2 in [2] it then follows that $P_M^R(t)$ is rational for all R and all M.

REFERENCES

[1] E. S. GOLOD (1962) – *On the homology of some local rings*, « Soviet Math. Dokl. », *3*.

[2] T. H. GULLIKSEN (1972) – *Massey operations and the Poincaré series of certain local rings*, « J. Algebra », *22*.

[3] T. H. GULLIKSEN – *A change of ring theorem with applications to Poincaré series and intersection multiplicity*. To appear in « Math. Scand. ».

[4] G. LEVIN – *Local rings and Golod homomorphisms*. To appear.

[5] J. SHAMASH (1969) – *The Poincaré series of a local ring*, « J. Algebra », *12*.

[6] G. SCHEJA (1964) – *Über die Bettizahlen lokaler Ringe*, « Math. Ann. », *155*.

[7] J. TATE (1957) – *Homology of noetherian rings and local rings*, « Illinois J. Math. », *1*.