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A study on the large oscillation stability of trojan asteroids around libration points

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Astronomia. — *A study on the large oscillation stability of trojan asteroids around libration points.* Nota di VITTORIO BANFI, presentata [(*)] dal Corrisp. M. G. FRACASTORO.

RIASSUNTO. — L'Autore considera il problema del moto dei pianetini troiani, tenendo presente anche l'azione del pianeta Saturno, al fine di studiarne le grandi oscillazioni attorno al punto triangolare L_4, per confermare la non esistenza di oscillazioni periodiche, già dimostrata dall'Autore stesso nel caso di tre corpi ristretto (sole, Giove, pianetino).

1. INTRODUCTION

Many investigations on the motion of a trojan asteroid around libration points were made since 1959 till today. The astronomical problem has been analytically approached by means of the mathematical model relating to the three-body problem, in the general form as well as in the restricted one.

In a fundamental research carried on by W. H. Jefferys and J. Moser (1966), a rigorous proof of the existence of a class of quasi-periodic solutions of the three-body problem is given. Approximate solutions of the motion problem, especially for astronomic and astrometric applications, are also very important: they were found by P. I. Message (1959), E. Rabe (1961–1962) and E. F. Goodrich (1966). In these studies the mentioned scientists endeavoured to force the solution of the three-body problem into the frame of Fourier series.

The present study intends to approach the trojan asteroid motion problem, starting from a more complicate mathematical model in respect of that relating to the normal restricted three-body problem. In fact, the possible perturbing action of Saturn on the asteroid motion is considered in order to find, if it exists, an influence on stability or on instability of the oscillations. Geometrical and dynamical problems are established through the two following steps:

1) restricted four-body problem, suitably defined;
2) large oscillation problem investigation (around libration points) in order to confirm the non-existence of period oscillations, in the restricted three-body problem, previously proved (Banfi, 1973).

2. FUNDAMENTAL ASSUMPTIONS OF THE MODEL

Saturn orbit has a small eccentricity ($e = 0.0557$) as well as Jupiter ($e = 0.0484$); therefore, we suppose the two orbits be circular and in the same invariable plane which contains also the Sun and the trojan asteroids. Let

(*) Nella seduta dell'8 febbraio 1975.

O be the common centre of mass of the complete system, the origin of a rectangular plane co-ordinate frame of reference. If we call $(P_i - O)$ the radius vectors, corresponding to the three principal mass-points (i.e. $i = 1$ Sun, $i = 2$ Jupiter, $i = 3$ Saturn), and m_i the respective masses, then

$$(a) \qquad \sum_{1}^{3}{}_i (P_i - O)\, m_i = 0$$

in every instant of the motion. Each one of the three radius vectors has its own angular velocity around O (fixed point) and also constant amplitude; furthermore, from the relation (a), we have:

$$(b) \qquad (P_1 - O) + (P_2 - O)\frac{m_2}{m_1} + (P_3 - O)\frac{m_3}{m_1} = 0\,,$$

where m_2/m_1 and m_3/m_1 are defined in this way

$$\frac{m_2}{m_1} = \frac{\text{Jupiter mass}}{\text{Sun mass}} = 0.955 \cdot 10^{-3} = \mu\,;$$

$$\frac{m_3}{m_1} = \frac{\text{Saturn mass}}{\text{Sun mass}} = 0.286 \cdot 10^{-3} = 0.32\,\mu$$

and also

$$\frac{|P_3 - O|}{|P_2 - O|} = 1.81\,.$$

Besides, each radius vector $(P_i - O)$ can be analitically expressed in the following manner

$$(P_i - O) = |P_i - O|\, \bar{k}_i$$

whereas $\bar{k}_i$ is the unit vector having the direction of $(P_i - o)$. Relation (b) takes the form

$$(P_1 - O) = -|P_2 - O| \cdot 0.955 \cdot 10^{-3}\, \bar{k}_2 - |P_3 - O| \cdot 0.286 \cdot 10^{-3}\, \bar{k}_3$$

or

$$(P_3 - O) = -|P_3 - O| \cdot 0.286 \cdot 10^{-3}\, \bar{k}_3 - |P_2 - O| \cdot 0.955 \cdot 10^{-3}\, \bar{k}_2\,.$$

If the rectangular plane co-ordinate system, instead of having the origin in the common centre of mass, had P_1 as origin, then the new radius vectors $(P_2 - P_1)$ and $(P_3 - P_1)$ should be

$$(P_2 - P_1) = (P_2 - O) + (O - P_1) =$$
$$= |P_2 - O|\,(0.955 \cdot 10^{-3}\, \bar{k}_2 + 1.81 \cdot 0.286 \cdot 10^{-3}\, \bar{k}_3) + (P_2 - O)$$

$$(P_3 - P_1) = (P_3 - O) + (O - P_1) =$$
$$= |P_3 - O| \left(0.286 \cdot 10^{-3}\, \bar{k}_3 + \frac{0.955}{1.81} \cdot 10^{-3}\, \bar{k}_2\right) + (P_3 - O)\,.$$

In these relations, it is pointed out that the vectors $(P_2 - P_1)$ and $(P_3 - P_1)$ differ from $(P_2 - O)$ and $(P_3 - O)$, in each instant of the motion, by for two very small vectors whose magnitude is about $1/1000$ of the magnitude of the same vectors (P_2 — O) and (P_3 — O). Then the error deriving from the assumptions

a) origin in P_1;

b) circular orbits, for P_2 and P_3, with P_1 as centre,

is smaller than that relating to the initial assumption, i.e. circularity and complanarity of Jupiter and Saturn orbits. In any case, this error does not invalidate the final goal of the present study.

3. Analytical development of the model

By means of the previous assumptions, the restricted four-body problem is established in a way similar that relating to the restricted three-body to problem. Again, the mass of the trojan asteroid is considered so small to have no gravitational influence on the three principal bodies (Sun, Jupiter and Saturn), and at the same time it is influenced by those.

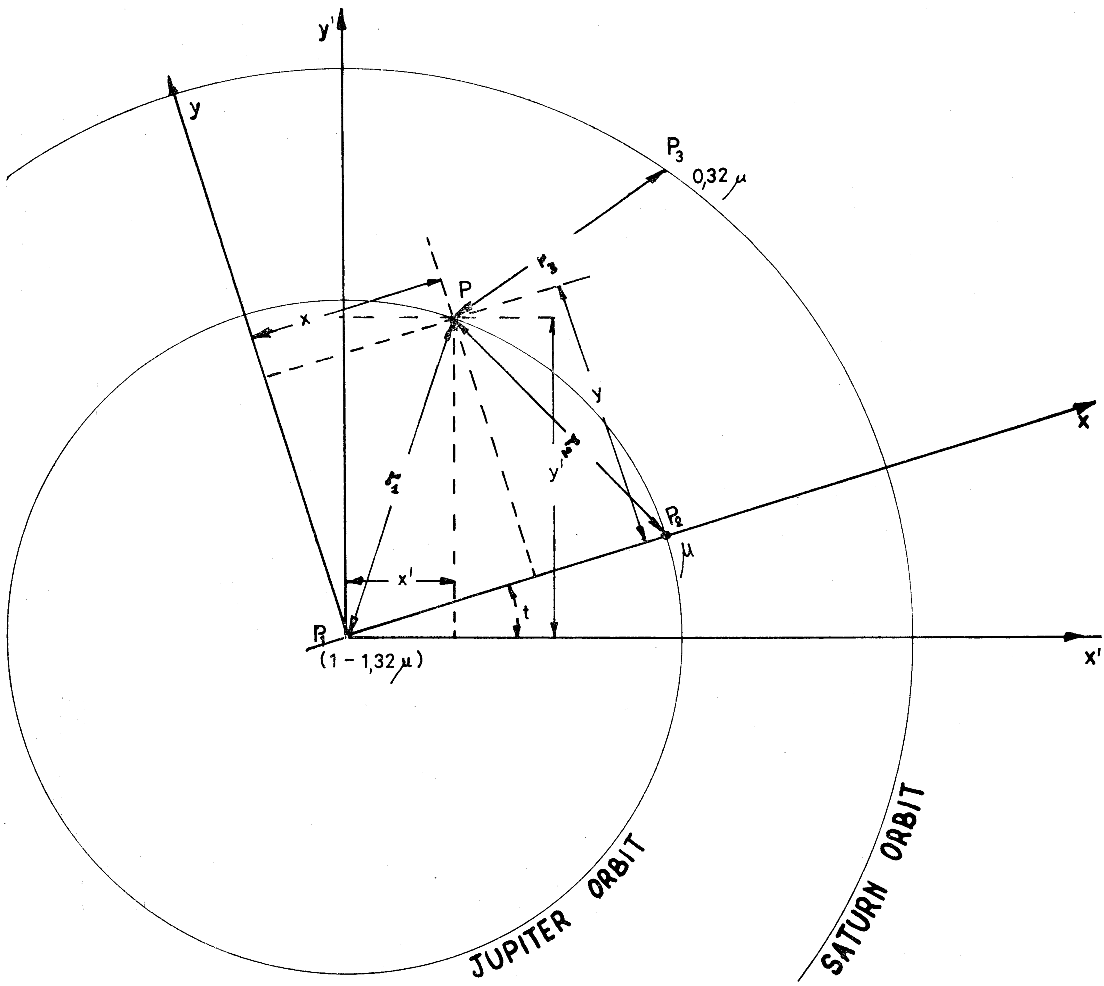

Fig. 1. – S′ (x', y') and S (x, y) co-ordinate systems. P infinitesimal mass asteroid.

Let us assume a rectangular plane co-ordinate system $S'(x', y')$ with P_1 as origin (fig. 1), following the standard method of the restricted three-body problem (Finlay-Freundlich, 1958), and P_2, P_3 moving with uniform speed in two circles around the origin P_1 as common centre. Now we change from a fixed co-ordinate system to another system $S(x, y)$ with the following properties. The new x-axis passes through P_1 and P_2, while the origin is still P_1, and obviously it is rotated (with the angular velocity of Jupiter around the Sun) with respect to the x'-axis of the preceding system. The problem of the motion of the asteroid is transformed into the problem of the motion of a body of infinitesimal mass under the attraction of two fixed mass-points P_1 and P_2, both lying on the x-axis, and of a third mass-point P_3 which moves with circular motion around the centre P_1. The final stage is to obtain a system of two differential equations, with the new (x, y) variables, for suitable discussion. At first, it is convenient to assume the standard form of the equations (Finlay-Freundlich 1958), i.e. without the superfluous constants. Introducing the two ratios

$$\mu = \frac{m_2}{m_1 + m_2 + m_3} \cong \frac{m_2}{m_1} \qquad \frac{m_3}{m_1 + m_2 + m_3} \cong \frac{m_3}{m_1} = 0.32\,\mu$$

and also assuming $|P_2 - P_1| = 1$ and $|P_3 - P_1| = 1.81$, and a suitable time-scale in order to render the constant of gravitation also equal to unity, we can write the equations of motion of the asteroid with respect to $S'(x', y')$ system:

$$(1) \qquad \begin{cases} \dfrac{d^2 x'}{dt^2} = -(1 - 1.32\,\mu)\,\dfrac{x'}{r_1^3} - \mu\,\dfrac{x' - x_2'}{r_2^3} - 0.32\,\mu\,\dfrac{x' - x_3}{r_3^3} \\[2ex] \dfrac{d^2 y'}{dt^2} = -(1 - 1.32\,\mu)\,\dfrac{y'}{r_1^3} - \mu\,\dfrac{y' - y_2'}{r_2^3} - 0.32\,\mu\,\dfrac{y' - y_3'}{r_3^3} \end{cases}$$

where r_1, r_2 and r_3 denote the distances of the asteroid from the three finite masses. Changing reference system, i.e. from $S'(x', y')$ to $S(x, y)$ by means of the formulae of transformation

$$(2) \qquad \begin{aligned} x' &= x \cos t - y \sin t \\ y' &= y \cos t + x \sin t \end{aligned}$$

we get the equations

$$(3) \qquad \begin{cases} \dfrac{d^2 x'}{dt^2} = \left(\dfrac{d^2 x}{dt^2} - 2\,\dfrac{dy}{dt} - x\right) \cos t - \left(\dfrac{d^2 y}{dt^2} + 2\,\dfrac{dx}{dt} - y\right) \sin t \\[2ex] \dfrac{d^2 y'}{dt^2} = \left(\dfrac{d^2 y}{dt^2} + 2\,\dfrac{dx}{dt} - y\right) \cos t + \left(\dfrac{d^2 x}{dt^2} - 2\,\dfrac{dy}{dt} - x\right) \sin t. \end{cases}$$

The co-ordinates of P_2 and P_3 in respect of $S'(x', y')$ system are

$$(2\text{ bis}) \qquad P_2 \begin{cases} x_2' = \cos t \\ y_2' = \sin t \end{cases} \qquad P_3 \begin{cases} x_3' = 1.81 \cos 0{,}408\, t \\ y_3' = 1.81 \sin 0{,}408\, t\,. \end{cases}$$

Substituting into (1) the relations (2), (2bis) and (3), it is found that the differential equation system, in (x, y) co-ordinates, is

$$
(4)\quad \left\{
\begin{aligned}
&\left[\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - x + (1 - 1.32\,\mu)\frac{x}{r_1^3} + \mu\frac{x-1}{r_2^3} + 0.32\,\mu\frac{x}{r_3^3}\right]\cos t - \\
&\quad - \left[\frac{d^2y}{dt^2} + 2\frac{dx}{dt} - y + (1 - 1.32\,\mu)\frac{y}{r_1^3} + \frac{\mu y}{r_2^3} + 0.32\,\mu\frac{y}{r_3^3}\right]\sin t - \\
&\quad - 0.32 \cdot 1.81 \cdot \mu\frac{\cos 0.408\,t}{r_3^3} = 0 \\
&\left[\frac{d^2y}{dt^2} + 2\frac{dx}{dt} - y + (1 - 1.32\,\mu)\frac{x}{r_1^3} + \frac{\mu y}{r_2^3} + 0.32\frac{\mu y}{r_3^3}\right]\cos t + \\
&\quad + \left[\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - x + (1 - 1.32\,\mu)\frac{x}{r_1^3} + \frac{\mu(x-1)}{r_2^3} + 0.32\frac{\mu x}{r_3^3}\right]\sin t - \\
&\quad - 0.32 \cdot 1.81 \cdot \mu\frac{\sin 0.408\,t}{r_3^3}\,.
\end{aligned}
\right.
$$

From these two equations the final differential equations in x and y are obtained after multiplying by $\cos t$ and $\sin t$ and adding and subtracting respectively; they are

$$
(5)\quad \left\{
\begin{aligned}
&\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = x - (1 - 1.32\,\mu)\frac{x}{r_1^3} - \frac{\mu(x-1)}{r_2^3} - 0.32\,\mu\frac{x}{r_3^3} + \\
&\quad + 0.32 \cdot 1.81\frac{\mu}{r_3^3}\cos 0{,}592\,t \\
&\frac{d^2y}{dt^2} + 2\frac{dx}{dt} = y - (1 - 1.32\,\mu)\frac{y}{r_1^3} - \frac{\mu y}{r_2^3} - 0.32\,\mu\frac{y}{r_3^3} - \\
&\quad - 0.32 \cdot 1.81\frac{\mu}{r_3^3}\sin 0{,}592\,t\,.
\end{aligned}
\right.
$$

Remembering that r_1, r_2 and r_3 can be written in the form:

$$r_1 = \sqrt{x^2 + y^2}$$

$$r_2 = \sqrt{(x-1)^2 + y^2}$$

$$r_3 = \sqrt{(x - 1.81\cos 0.592\,t)^2 + (y - 1.81\sin 0.592\,t)^2}$$

the system (5) becomes:

$$
(6)\quad \begin{cases}
\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = x - (1 - 1.32\,\mu)\dfrac{x}{(x^2+y^2)^{3/2}} - \mu\dfrac{x-1}{[(x-1)^2+y^2]^{3/2}} - \\
\quad - 0.32\,\mu\dfrac{x}{[(x-1.81\cos 0.592\,t)^2+(y-1.81\sin 0.592\,t)^2]^{3/2}} + \\
\quad + 0.32\cdot 1.81\,\mu\dfrac{\cos 0.592\,t}{[(x-1.81\cos 0.592\,t)^2+(y-1.81\sin 0.592\,t)^2]^{3/2}} \\
\dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = y - (1 - 1.32\,\mu)\dfrac{y}{(x^2+y^2)^{3/2}} - \mu\dfrac{y}{[(x-1)^2+y^2]^{3/2}} - \\
\quad - 0.32\,\mu\dfrac{y}{[(x-1.81\cos 0.592\,t)^2+(y-1.81\sin 0.592\,t)^2]^{3/2}} - \\
\quad - 0.32\cdot 1.81\,\mu\dfrac{\sin 0.592\,t}{[(x-1.81\cos 0.592\,t)^2+(y-1.81\sin 0.592\,t)^2]^{3/2}}\,.
\end{cases}
$$

The differential equation system (6) describes analytically the motion of the asteroid P, with respect to the system $S(x, y)$, under the attraction of the Sun and Jupiter and under the perturbing force exerted by the planet Saturn. Let $x = x_0 = 1/2$, $y = y_0 = \sqrt{3}/2$ denote the co-ordinates of a point of libration; let also substitute, in system (6), x and y with the expression:

$$x = x_0 + \xi = \xi + \frac{1}{2} \quad , \quad y = y_0 + \eta = \eta + \frac{\sqrt{3}}{2}\,;$$

then it is possible to study the motion of P, in terms of co-ordinates ξ and η, i.e. to investigate, if they exist, large and stable oscillations arount libration point $P_0(x_0, y_0)$. In other words, the problem is to look for periodic or non-periodic solutions of system (6), i.e. large and stable, or unstable, oscillations starting from libration point.

Putting the 6 (bis) into the system (6), we obtain:

$$
(7)\quad \begin{cases}
\dfrac{d^2\xi}{dt^2} - 2\dfrac{d\eta}{dt} = \xi + \dfrac{1}{2} - (1 - 1.32\mu)\dfrac{\xi + \frac{1}{2}}{\left[\left(\xi+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}} - \mu\dfrac{\xi - \frac{1}{2}}{\left[\left(\xi-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}} \\
\quad - 0.32\,\mu\dfrac{\xi+\frac{1}{2}}{\left[\left(\xi+\frac{1}{2}-1.81\cos 0.592\,t\right)^2+\left(\frac{\sqrt{3}}{2}+\eta-1.81\sin 0.592\,t\right)^2\right]^{3/2}} + \\
\quad + 0.32\cdot 1.81\cdot\mu\dfrac{\cos 0.592\,t}{\left[\left(\xi+\frac{1}{2}-1.81\cos 0.592\,t\right)^2+\left(\frac{\sqrt{3}}{2}+\eta-1.81\sin 0.592\,t\right)^2\right]^{3/2}} \\
\dfrac{d^2\eta}{dt^2} + 2\dfrac{d\xi}{dt} = \eta + \dfrac{\sqrt{3}}{2} - (1 - 1.32\,\mu)\dfrac{\eta + \frac{\sqrt{3}}{2}}{\left[\left(\xi+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}} - \mu\dfrac{\eta + \frac{\sqrt{3}}{2}}{\left[\left(\xi-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}} - \\
\quad - 0{,}32\,\mu\dfrac{\frac{\sqrt{3}}{2}+\eta}{\left[\left(\xi+\frac{1}{2}-1.81\cos 0.592\,t\right)^2+\left(\eta+\frac{\sqrt{3}}{2}-1.81\sin 0.592\,t\right)^2\right]^{3/2}} - \\
\quad - 0.32\cdot 1.81\cdot\mu\dfrac{\sin 0.592\,t}{\left[\left(\xi+\frac{1}{2}-1.81\cos 0.592\,t\right)^2+\left(\eta+\frac{\sqrt{3}}{2}-1.81\sin 0.592\,t\right)^2\right]^{3/2}}\,.
\end{cases}
$$

In a more compact form, the above system becomes:

$$(8)\qquad \begin{cases} \dfrac{d^2\xi}{dt^2} - 2\dfrac{d\eta}{dt} = A(\xi,\eta) \\ \dfrac{d^2\eta}{dt^2} + 2\dfrac{d\xi}{dt} = B(\xi,\eta) \end{cases}$$

where the functions A and B can be extracted from the system (7) with simple analytical work. Starting from a previous paper (Banfi 1973), the knowledge of periodicity or non-periodicity, of eventual solutions $\xi(t)$ and $\eta(t)$ of system (8), is obtained by the study of tne indicating function $F = \frac{\partial A}{\partial \xi} + \frac{\partial B}{\partial \eta}$.

A simply connected region in the plane (ξ,η), obviously including libration point P_0, where the function F does not change sign in every instant of time belonging to the full period of a complete revolution of P_3 around P_1, contains no closed trajectories. In this case, no periodic solutions of equation (8) exist.

Let us write down the analytical expression for $F = \frac{\partial A}{\partial \xi} + \frac{\partial B}{\partial \eta}$. The calculation gives

$$(9)\qquad F = 2 + (1 - 1.32\mu)\frac{1}{\left[\left(\xi+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}} + \mu\frac{1}{\left[\left(\xi-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}}$$

$$- 0.64\,\mu\,\frac{1}{(\Delta_1)^{3/2}} + 3\cdot 0.32\,\mu\,\frac{\left(\frac{\sqrt{3}}{2}+\eta\right)\Delta_3 + \left(\xi+\frac{1}{2}\right)\Delta_2}{\Delta_1^{5/2}} +$$

$$+ 3\cdot 0.32\cdot\mu\cdot 1.81\,\frac{(\sin 0.592\,t)\,\Delta_3 - (\cos 0.592\,t)\,\Delta_2}{\Delta_1^{5/2}}$$

where

$$\Delta_1 = \left(\xi + \frac{1}{2} - 1.81\cos 0.592\,t\right)^2 + \left(\eta + \frac{\sqrt{3}}{2} - 1.81\sin 0.592\,t\right)^2$$

$$\Delta_2 = \xi + \frac{1}{2} - 1.81\cos 0.592\,t$$

$$\Delta_3 = \eta + \frac{\sqrt{3}}{2} - 1.81\sin 0.592\,t\,.$$

Let us examine the values of F in an established region including the libration point P_0. A closed curve is drawn in fig. 2 which contains the largest astronomically observed oscillations of trojan asteroids. It is convenient to fix a definite field, for numerical verification of the function F which includes the closed curve mentioned above. Choosing then each value for η of this set

$$+0.2,\quad +0.1,\quad 0,\quad -0.1,\quad -0.2,\quad -0.3,\quad -0.4$$

it is possible to calculate the second and third term of F [formula (9)] in correspondence of each value for ξ which allows to check completely the

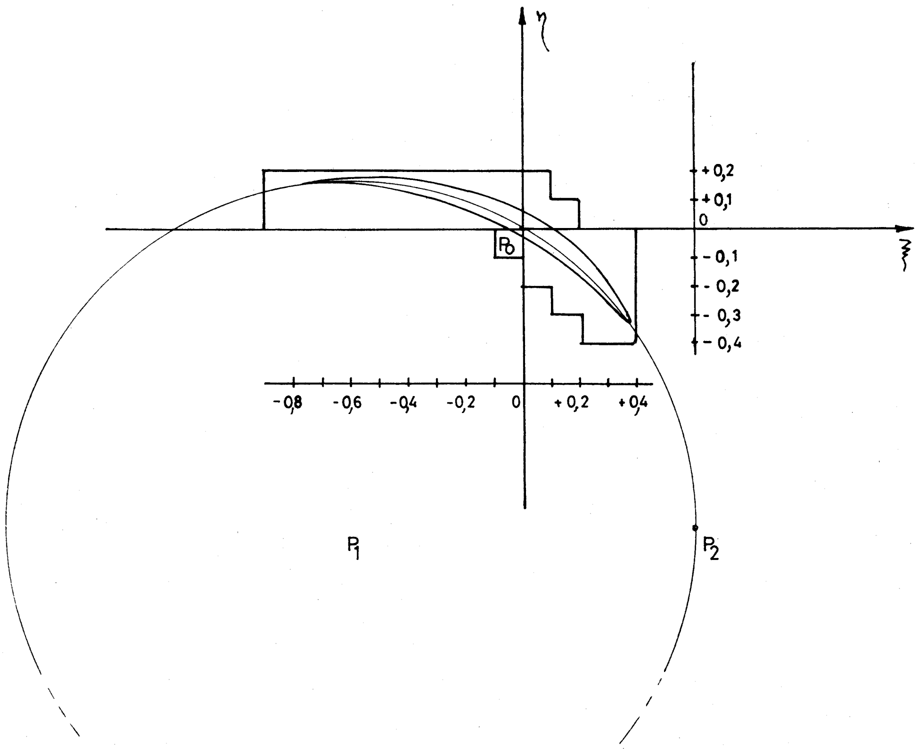

Fig. 2. – Linear squared region for numerical check; curved region containing observed oscillations.

region established in fig. 2. The remainder terms of F depend also upon t, through Δ_1, Δ_2 and Δ_3. Writing F in the following form

$$F = K_0 - K_1$$

where K_0 and K_1 are given by the formulae:

$$K_0 = 2 + (1 - 1.32\mu)\frac{1}{\left[\left(\xi+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}} + \mu\frac{1}{\left[\left(\xi-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}+\eta\right)^2\right]^{3/2}}$$

$$K_1 = 0.64\,\mu\,\frac{1}{\Delta_1^{3/2}} -$$

$$-\mu\,\frac{0.96\left[\left(\frac{\sqrt{3}}{2}+\eta\right)\Delta_3+\left(\xi+\frac{1}{2}\right)\Delta_2\right]+1.74\,[(\sin 0.592\,t)\,\Delta_3-(\cos 0.592\,t)\Delta_2]}{\Delta_1^{5/2}}$$

because K_0 is positive, it is necessary to calculate, within the field established above and during the period of the revolution time of P_3, the maximum positive value of K_1. This maximization calculus was carried out by an electronic computer. Numerical results are gathered in the enclosed Table I. It is easy to see that F remains always positive.

TABLE I.

η	ξ	F	η	ξ	F
+0,2	—0,8	2,7238	0	—0,3	3,4189
+0,2	—0,7	2,7722	0	—0,2	3,2936
+0,2	—0,6	2,8038	0	—0,1	3,1464
+0,2	—0,5	2,8148	0	0	2,9939
+0,2	—0,4	2,8039	0	+0,1	2,8481
+0,2	—0,3	2,7724	0	+0,2	2,7156
+0,2	—0,2	2,7241	0	+0,3	2,5987
+0,2	—0,1	2,6636	0	+0,4	2,4965
+0,2	0	2,5961	—0,1	—0,1	3,5454
+0,2	+0,1	2,5256	—0,1	0	3,3022
+0,1	—0,8	2,9582	—0,1	+0,1	3,0811
+0,1	—0,7	3,0341	—0,1	+0,2	2,8900
+0,1	—0,6	3,0845	—0,1	+0,3	2,7296
+0,1	—0,5	3,1023	—0,1	+0,4	2,5967
+0,1	—0,4	3,0847	—0,2	0	3,7281
+0,1	—0,3	3,0343	—0,2	+0,1	3,3855
+0,1	—0,2	2,9586	—0,2	+0,2	3,1060
+0,1	—0,1	2,8665	—0,2	+0,3	2,8835
+0,1	0	2,7674	—0,2	+0,4	2,7082
+0,1	+0,1	2,6683	—0,3	+0,1	3,7804
+0,1	+0,2	2,5740	—0,3	+0,2	3,3701
0	—0,8	3,2932	—0,3	+0,3	3,0624
0	—0,7	3,4186	—0,3	+0,4	2,8318
0	—0,6	3,5039	—0,4	+0,2	3,6833
0	—0,5	3,5342	—0,4	+0,3	3,2635
0	—0,4	3,5041	—0,4	+0,4	2,9652

4. CONCLUSIONS

The analytical development of the proposed model shows that the non-existence, generally speaking and for the examined region, of periodic solutions can be affirmed in the case of large oscillations (in the plane ξ, η) around the libration point, of the trojan asteroid. The region under consideration includes largely the maximum deviations astronomically observed.

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