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## On wave solutions of the field equations of general relativity containing electromagnetic fields in a generalized Peres-space-time

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**Relatività.** — *On wave solutions of the field equations of general relativity containing electromagnetic fields in a generalized Peres-space-time.* Nota di KRISHNA BIHARI LAL e ANIRUDH PRADHAN, presentata (*) dal Socio C. CATTANEO.

RIASSUNTO. — Soluzioni del tipo onda piana per le equazioni di campo della relatività generale e della teoria del campo unificato non simmetrico sono state studiate da Takeno nello spazio-tempo di Peres. La metrica spazio-temporale di Peres è stata in seguito generalizzata da Lal e Pandey i quali hanno trovato per essi soluzioni del tipo onda piana delle equazioni di campo della relatività generale. In questa Nota noi abbiamo considerato un'altra generalizzazione dello spazio-tempo di Peres stabilendo l'esistenza di soluzioni del tipo onda piana per le equazioni di campo della relatività generale.

## 1. INTRODUCTION

A Peres [1] has found out an exact plane wave-like solution of the Einstein's field equation for empty region

$$K_{ij} = 0 , \qquad (i , j = 1 , \cdots , 4) \tag{1.1}$$

in the space-time

$$ds^2 = - dx^2 - dy^2 - dz^2 + dt^2 - 2f(x , y , z + t)(dz + dt)^2 , \tag{1.2}$$

where $K_{ij}$ is the Ricci tensor of the space-time.

Later on H. Takeno [2], [3] found out the plane wave-like solutions of the field equations of general relativity and non-symmetric unified field theory in (1.2).

Lal and Pandey [4] have generalized the line element (1.2) as

$$ds^2 = - A\, dx^2 - B\, dy^2 - (1 - E)\, dz^2 - 2\, E\, dz\, dt + (1 + E)\, dt^2 , \tag{1.3}$$

where A and B are functions of $z$ and $t$ and $E = E(x , y , z , t)$. These authors have also found out the plane wave-like solutions of the field equations of general relativity.

In this paper we have considered another generalization of Peres-space-time (1.2) by taking the metric as

$$ds^2 = - A\, dx^2 - B\, dy^2 - (1 - E)\, dz^2 - 2\, E\, dz dt + (1 + E)\, dt^2 , \tag{1.4}$$

where A, B and E are all taken as functions of $x , y$ and $Z (= z - t)$.

In this new generalization also the metric (1.4) does not wholly conform to Takeno's definition [5] of the plane wave because although the first criter-

(*) Nella seduta dell'8 febbraio 1975.

ion of Takeno's plane wave is satisfied, the second criterion is not satisfied since the components of the metric tensor in our case are functions of $(x, y, Z)$ and not of Z alone. We shall, therefore, find the plane wave-like solutions of the field equations of general relativity

$$K_{ij} = -8\pi E_{ij}, \tag{1.5}$$

where $E_{ij}$ is the electromagnetic energy tensor defined by

$$E_{ij} = \frac{1}{4} g_{ij} F^{lm} F_{lm} - F_{il} F_{jm} g^{lm}. \tag{1.6}$$

Here $g_{ij}$ is the fundamental tensor of the space-time (1.4) and $F_{ij}$ is the anti-symmetric electromagnetic field tensor satisfying the generalized Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \tag{1.7}$$

$$F^{ij}_{;k} = 0. \tag{1.8}$$

A semicolon and a comma followed by an index denote covariant and partial differentiation respectively.

## 2. Calculation of Christoffel symbols and the Ricci tensor

The non-vanishing components of the contravariant tensor $g^{ij}$ made from (1.4) are

$$\begin{aligned} &g^{11} = -1/A \quad , \quad g^{22} = -1/B \quad , \quad g^{33} = -(1 + E), \\ &g^{44} = 1 - E \quad , \quad g^{34} = g^{43} = -E, \end{aligned} \tag{2.2}$$

while the components of the Christoffel symbols of the second kind are

$$\left\{\begin{aligned}
&\left\{ {k \atop 11} \right\} = \left[ \tfrac{1}{2} A_x/A \;,\; -\tfrac{1}{2} A_y/B \;,\; -\tfrac{1}{2}\bar{A} \;,\; -\tfrac{1}{2}\bar{A} \right], \\
&\left\{ {k \atop 12} \right\} = \left[ \tfrac{1}{2} A_y/A \;,\; \tfrac{1}{2} B_x/B \;,\; 0 \;,\; 0 \right], \\
&\left\{ {k \atop 22} \right\} = \left[ -\tfrac{1}{2} B_x/A \;,\; \tfrac{1}{2} B_y/B \;,\; -\tfrac{1}{2}\bar{B} \;,\; -\tfrac{1}{2}\bar{B} \right], \\
&\left\{ {k \atop 13} \right\} = -\left\{ {k \atop 14} \right\} = \left[ \tfrac{1}{2}\bar{A}/A \;,\; 0 \;,\; -\tfrac{1}{2} E_x \;,\; -\tfrac{1}{2} E_x \right], \\
&\left\{ {k \atop 23} \right\} = -\left\{ {k \atop 24} \right\} = \left[ 0 \;,\; \tfrac{1}{2}\bar{B}/B \;,\; -\tfrac{1}{2} E_y \;,\; -\tfrac{1}{2} E_y \right], \\
&\left\{ {k \atop 33} \right\} = -\left\{ {k \atop 34} \right\} = \left\{ {k \atop 44} \right\} = \left[ \tfrac{1}{2} E_x/A \;,\; \tfrac{1}{2} E_y/B \;,\; -\tfrac{1}{2}\bar{E} \;,\; -\tfrac{1}{2}\bar{E} \right].
\end{aligned}\right. \tag{2.2}$$

Here and it what follows, the lower suffixes $x$ and $y$ attached with any function indicate ordinary partial differentiation with respect to $x$ and $y$ respectively, while a single and double bars over them denote respectively first and second partial derivatives with respect to Z.

The non-vanishing components of the curvature tensor $K_{ijlm}$ ($= - K_{jiml} = - K_{ijml} = K_{lmij}$) are given by

$$K_{1212} = L \quad , \quad K_{1213} = - K_{1214} = M \quad , \quad K_{1223} = - K_{1224} = N \; ,$$

(2.3) $$K_{1313} = - K_{1314} = K_{1414} = \alpha \; , \quad K_{1323} = - K_{1324} = - K_{1423} = K_{1424} = \beta$$

$$K_{2323} = - K_{2324} = K_{2424} = \eta \; ,$$

where

(2.4)
$$\begin{cases} 2\,L = A_{yy} + B_{xx} - \frac{1}{2}\,(A_y^2 + A_x B_x)/A - \frac{1}{2}\,(B_x^2 + A_y B_y)/B \; , \\ 2\,M = \bar{A}_y - \frac{1}{2}\,A_y\,(\bar{A}/A + \bar{B}/B) \; , \\ 2\,N = - \bar{B}_x + \frac{1}{2}\,B_x\,(\bar{A}/A + \bar{B}/B) \; , \\ 2\,\alpha = \bar{\bar{A}} - E_{xx} - \frac{1}{2}\,(\bar{A}^2 - A_x E_x)/A - \frac{1}{2}\,A_y E_y/B \; , \\ 2\,\beta = - E_{xy} + \frac{1}{2}\,(A_y A_x/A + B_x E_y/B) \; , \\ 2\,\eta = \bar{\bar{B}} - E_{yy} - \frac{1}{2}\,(\bar{B}^2 - B_y E_y)/B - \frac{1}{2}\,B_x E_x/A \; . \end{cases}$$

The surviving components of the Ricci tensor $K_{ij}$ obtained from (2.3) on contraction with the help of (2.1) are given by

(2.5) $$K_{11} = L/B \quad , \quad K_{13} = - K_{14} = - N/B \quad , \quad K_{22} = L/A \; ,$$

$$K_{23} = - K_{24} = M/A \; , \quad K_{33} = - K_{34} = K_{44} = \alpha/A + \eta/B \; .$$

## 3. THE ELECTROMAGNETIC FIELD

Following the assumptions and the method of calculation of Lal and Pandey [4], the components of the electromagnetic field $F_{ij}$ are given by

(3.1) $$F_{ij} = \begin{bmatrix} 0 & 0 & -\sigma & \sigma \\ 0 & 0 & \rho & -\rho \\ \sigma & -\rho & 0 & 0 \\ -\sigma & \rho & 0 & 0 \end{bmatrix}$$

with the condition

(3.2) $$\rho_x + \sigma_y = 0 \; ,$$

where $\rho$ and $\sigma$ are some functions of $(x, y, Z)$. The contravariant components of the electromagnetic field tensor $F^{ij}$ are given by

$$F^{ij} = \begin{bmatrix} 0 & 0 & -\sigma/A & -\sigma/A \\ 0 & 0 & \rho/B & \rho/B \\ \sigma/A & -\rho/B & 0 & 0 \\ \sigma/A & -\rho/B & 0 & 0 \end{bmatrix} . \tag{3.3}$$

Then we can easily show that $F_{ij}$ is null i.e.

$$F_{ij} F^{ij} = 0 \quad , \quad F_{ij} F_{kl} E^{ijkl} = 0 , \tag{3.4}$$

where $E^{ijkl}$ is the tensor whose components are antisymmetric with respect to each pair of indices and $E_{1234} = \sqrt{-g} = \sqrt{AB}$ , $g = \det. (g_{ij})$.

Using (2.1), (3.1) and (3.3) in (1.6), we find that the surviving components of electromagnetic energy tensor are given by

$$E_{33} = - E_{34} = - E_{43} = E_{44} = (A\rho^2 + B\sigma^2)/AB . \tag{3.5}$$

## 4. Solution of the field equations (1.5), (1.7) and (1.8)

Substituting the components of $K_{ij}$ and $E_{ij}$ from (2.5) and (3.5) respectively in the field equation (1.5) we obtain

$$L = 0 , \tag{4.1}$$

$$M = 0 , \tag{4.2}$$

$$N = 0 , \tag{4.3}$$

$$B\alpha + A\eta = - 8 \pi (A\rho^2 + B\sigma^2) , \tag{4.4}$$

where $L , M , N , \alpha , \eta$ are previously stated in (2.4).

Equations (4.1), (4.2) and (4.3), after integration reduce to

$$a) \; (B_x/\sqrt{AB})_x + (A_y/\sqrt{AB})_y = 0, \quad b) \; A_y/\sqrt{AB} = K_1 , \quad c) \; B_x/\sqrt{AB} = K_2 , \tag{4.5}$$

respectively, where $K_1$ and $K_2$ are arbitrary functions of $x$ and $y$. To obtain the values of A and B we assume the relation

$$A = Bf , \tag{4.6}$$

where $f$ is any function of Z alone.

From (4.5) *b*) and *c*) we get

$$B_x A_y = A B K_1 K_2 . \tag{4.7}$$

With the help of (4.6), equation (4.7) reduces to

$$pq = \psi , \qquad (p = B_x , \; q = B_y , \; \psi = K_1 K_2 B^2) . \tag{4.8}$$

Equation (4.8) is a standard form of a differential equation whose solution can be obtained by using Charpit's method [6].

For finding the value of E, we use (4.6) in (4.4), which on simplification gives

$$(4.9) \qquad E_{xx}/f + E_{yy} = 2\bar{\bar{B}} - (\bar{B})^2/B + \bar{B}\bar{f}/f + B\bar{\bar{f}}/f - \frac{1}{2}(\bar{f}/f)^2 B + 16\pi(\rho^2 + \sigma^2/f).$$

In view of (3.1) Maxwell's first field equation (1.7) is identically satisfied, while in view of (3.3) and (4.6) Maxwell's second field equation (1.8) gives the condition

$$(4.10) \qquad \sigma_x - f\rho_y = 0 .$$

Thus the plane wave-like solutions of the field equations (1.5), (1.7) and (1.8) are composed of $g_{ij}$ satisfying the condition (4.6) and $F_{ij}$ given by (3.1) with (3.2) under the conditions (4.5) *a*), (4.9) and (4.10).

It is worth noting that if $K_1$ and $K_2$ are two non-zero scalar constants, then from (4.8) the values of A, B satisfying equations (4.5) are given by

$$(4.11) \qquad \begin{aligned} \log B &= x\sqrt{f} + \lambda y/\sqrt{f} + \mu , \\ \log A &= x\sqrt{f} + \lambda y/\sqrt{f} + \log f + \mu , \end{aligned}$$

where $\mu$ and $\lambda$ are any scalar constants.

However if $K_1 = K_2 = 0$, then from (4.5) *b*) and *c*) we find that $A = A(x, Z)$ and $B = B(y, Z)$ in which case equations (4.1), (4.2) and (4.3) are identically satisfied and $g_{ij}$ satisfying

$$(4.12) \qquad A = A(x, Z) \quad , \quad B = B(y, Z) \quad , \quad E = E(x, y, Z) ,$$

and $F_{ij}$ given by (3.1) with (3.2) constitute the solutions of the field equations (1.5), (1.7) and (1.8) provided the conditions

$$(4.13) \qquad \{\bar{\bar{A}} - E_{xx} - (\bar{A}^2 - A_x E_x)/2A\}/A + \{\bar{\bar{B}} - E_{yy} - (\bar{B}^2 - B_y E_y)/2B\}/B = - 16\pi(\rho^2/B + \sigma^2/A) ,$$

and

$$(4.14) \qquad 2(A\rho_y - B\sigma_x) + (B\sigma/A) A_x - (A\rho/B) B_y = 0$$

hold.

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