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Examples Using Integral Equations to Determine Controllability

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Teoria dei controlli. — *Examples Using Integral Equations to Determine Controllability.* Nota di JERALD P. DAUER, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Nell'ipotesi che il sistema lineare $\dot{x} = A(t)x + B(t)u$ sia completamente controllabile, l'Autore riprende le sue ricerche sul sistema perturbato $\dot{x} = A(t)x + B(t)u + f(t, u)$ dove u soddisfa ad un'equazione integrale e dà condizioni utili per eliminare alcuna difficoltà connesse al suo problema.

In this paper several examples are presented which use integral equation methods to determine controllability properties of certain nonlinear systems. Throughout we will assume that the linear system

$$\dot{x} = A(t)x + B(t)u \tag{1}$$

is completely controllable; i.e. given any pair of points $x_0, x_1 \in E^n$ there exists a continuous control function $u : [0, T] \to E^m$, such that the solution of

$$\dot{x} = A(t)x + B(t)u(t)$$

$$x(0) = x_0$$

satisfies $x(T) = x_1$. Here A is a continuous $n \times n$ matrix function and B is a continuous $n \times m$ matrix function on $[0, T]$. Take $X(t)$ to be the fundamental matrix solution of $\dot{x} = A(t)x$ such that $X(0)$ is the identity matrix and let

$$W = \int_0^T X^{-1}(s)\, B(s)\, B(s)^*\, X^{-1}(s)^*\, ds,$$

where D^* denotes the transpose of D. Since system (1) is completely controllable, W is nonsingular [1].

In order to study the controllability of the perturbed system

$$\dot{x} = A(t)x + B(t)u + f(t, u) \tag{2}$$

we consider the nonlinear integral equation

$$u(t) = B(t)^*\, X^{-1}(t)^*\, W^{-1}\left[\bar{x} - \int_0^T X^{-1}(s)\, f(s, u(s))\, ds\right] \tag{3}$$

where $\bar{x} = X^{-1}(T)x_1 - x_0$. If the control function u is a solution of integral

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equation (3), then the corresponding solution of system (2) satisfying $x(0) = x_0$, $x(\mathrm{T}) = x_1$ is

$$x(t) = \mathrm{X}(t)\,x_0 + \mathrm{X}(t)\int_0^t \mathrm{X}^{-1}(s)\,[\mathrm{B}(s)\,u(s) + f(s, u(s))]\,\mathrm{d}s\,.$$

Hence the controllability of system (2) can be determined by solving equation (3). (A similar approach has been used by Vidyasagar [2] and Dauer [3]). It should be noted that a major difficulty in this approach is that X^{-1} and W^{-1} are, in general, unknown. Although for a specific system information about these may be available, most of our examples are chosen so as to eliminate or reduce this difficulty.

Define the kernel k and the function g by

$$g(t) = \mathrm{B}(t)^*\,\mathrm{X}^{-1}(t)^*\,\mathrm{W}^{-1}\bar{x},$$

$$k(t, s) = \mathrm{B}(t)^*\,\mathrm{X}^{-1}(t)^*\,\mathrm{W}^{-1}\,\mathrm{X}^{-1}(s)\,.$$

Then equation (3) can be written as the nonlinear Fredholm equation

$$u(t) + \int_0^{\mathrm{T}} k(t, s)\,f(s, u(s))\,\mathrm{d}s = g(t)\,. \tag{4}$$

Clearly the kernel k is symmetric if and only if B is a constant function.

Example 0. If f satisfies the condition

$$\lim_{|u|\to\infty} \frac{|f(t, u)|}{|u|} = 0\,,$$

uniformly for $t \in \mathrm{I}$, then system (2) is completely controllable. (Dauer [3] has generalized this result to systems where the functions A, B and f depend on (t, x, u)).

Example 1. Suppose $f(t, u) = \mathrm{C}(t)\,h(t, u)$ where $h(t, u)$ and u are real valued. Suppose $h(t, 0) \equiv 0$ and there are constants α, β restricting restricting the slope of h as follows:

$$\alpha \le \frac{h(t, u_1) - h(t, u_2)}{u_1 - u_2} \le \beta$$

for all $t \in \mathrm{I}$ and $u_1, u_2 \in \mathrm{E}^1$. For each constant c let

$$\eta(c) = \max\{|\beta - c|, |c - \alpha|\}\,,$$

and let L denote the linear integral operator with kernel $k(t, s)\,\mathrm{C}(s)$; i.e.,

$$(\mathrm{L}v)(t) = \int_{\mathrm{I}} k(t, s)\,\mathrm{C}(s)\,v(s)\,\mathrm{d}s\,.$$

The the system

$$\dot{x} = \mathrm{A}(t)\,x + \mathrm{B}(t)\,u + \mathrm{C}(t)\,h(t, u)$$

is completely controllable provided there is a number c which is not a characteristic value of L [4] and which satisfies the condition

$$\| (1 + cL^{-1}) \| \eta(c) < 1 ,$$

[5, Theorem III. 1.1]. In particular, since $c = 0$ is not a characteristic value of a linear operator, a sufficient condition is

$$\eta(0) = \max \{ | \beta | , | \alpha | \} < 1 .$$

Example 2. Suppose $B(t)$ is $n \times n$ and invertible for all $t \in I$. Since system (1) is completely controllable, W^{-1} is a positive definite, symmetric matrix. Hence there exists an invertible, symmetric matrix H such that $H^2 = W^{-1}$. Define the $n \times n$ matrix function α by

$$\alpha(t) = B(t)^* X^{-1}(t)^* H .$$

Then equation (4) can be written

$$u(t) + \alpha(t) \int_I \alpha(s)^* B^{-1}(s) f(s , u(s)) \, ds = \alpha(t) Hx .$$

Therefore, if u is a solution we can write

$$u(t) = \alpha(t) \xi ,$$

for some $\xi \in E^n$. Hence equation (4) becomes

$$\alpha(t) \xi + \alpha(t) \int_I \alpha(s)^* B^{-1}(s) f(s , \alpha(s) \xi) \, ds = \alpha(t) H\bar{x} .$$

Since $\alpha(t)$ is invertible this equation reduces to

$$\xi + \int_I \alpha(s)^* B^{-1}(s) f(s , \alpha(s) \xi) \, ds = H\bar{x} . \tag{5}$$

Then the integral equation (4) has a solution if and only if equation (5) can be solved for some $\xi \in E^n$. Therefore, there exists a control function u such that the corresponding solution of system (2) satisfies $x(0) = x_0$, $x(T) = x_1$ if and only if equation (5) has a solution $\xi \in E^n$. This control function will necessarily be of the form $u(t) = \alpha(t) \xi$. (For examples of perturbations f for which equation (5) can be solved see [6]).

Example 3. Suppose the control functions u are real valued and f is of the form $f(t , u) = C(t) u^m$ where m is any integer greater than 1. Then equation (4) has a solution provided $|\bar{x}|$ is sufficiently small [7, p. 154]. By setting $x_1 = 0$ we have that systems of the form

$$\dot{x} = A(t)x + B(t)u + C(t) u^m ,$$

$m > 1$ an integer, are locally controllable [8, p. 364].

Example 4. Suppose system (2) is a one dimensional system (i.e. $m = n = 1$) and that $B(t) \neq 0$ for all $t \in I$. Let $\eta > 0$ be the smallest characteristic value of the linear integral operator corresponding to the positive definite, symmetric kernel

$$k(t, s) = B(t) X^{-1}(t) W^{-1} X(s) B(s).$$

Suppose there exist continuous functions $c(t)$, $d(t)$ and constants a, γ, with $0 < \gamma < 2$, such that

$$(B(t))^{-1} \int_0^u f(t, u) \, du \leq \frac{a}{2} u^2 + b(t) \, | u |^{2-\gamma} + c(t),$$

for all $t \in I$, $u \in E^1$. If $a < \eta$, e.g. if $a \leq 0$, then system (2) is completely controllable [9, p. 307].

Example 5. Suppose B and C are square $n \times n$ matrices and that C is invertible. Let $\mathfrak{A}$ be the set of all real numbers d such that the system

$$\dot{x} = A(t) x + Bu + dCu \tag{6}$$

is completely controllable. It is known [10] that $\mathfrak{A}$ contains a neighborhood of the origin. In fact, $\mathfrak{A}$ contains all but at most a countable number of real numbers and those numbers not in $\mathfrak{A}$ do not have a finite limit point Therefore, $\mathfrak{A}$ is open and its closure is the set of all real numbers.

To see that $\mathfrak{A}$ has these properties let d be fixed. Using control functions of the form $v = Cu$ it follows that system (6) is equivalent to the system

$$\dot{x} = A(t) x + BC^{-1} v + dv.$$

If W_1 denotes the positive definite controllability matrix for the completely controllable system $\dot{x} = A(t) x + BC^{-1} v$, then equation (4) can be written as the linear integral equation

$$v(t) + d \int_I (BC^{-1})^* X^{-1}(t)^* W_1^{-1} X^{-1}(s) v(s) \, ds = g(t). \tag{7}$$

The Fredholm Alternative Theorem [4] shows that equation (7) has a solution for any continuous function g provided d is not a characteristic value for this integral operator. Since the kernel is symmetric the set $\mathfrak{A}$ has the desired properties [4, p. 87]. This result can be extended to almost linear perturbations by applying results of Krasnosel'skii [9, pp. 165–167].

REFERENCES

[1] R. E. KALMAN, Y. C. HO and K. S. NARENDRA (1963) – *Controllability of linear dynamical systems*, «Contributions to Differential Equations», *1*, 189–213.

[2] M. VIDYASAGAR (1972) – *A controllability condition for nonlinear systems*, «IEEE Transactions on Automatic Control», AC–*17*, 569–570.

[3] J. P. DAUER – *Nonlinear perturbations of quasi-linear control systems*, « Journal of Mathematical Analysis and Applications », to appear.

[4] J. A. COCHRAN (1972) – *The Analysis of Linear Integral Equations*, McGraw–Hill, New York.

[5] J. M. HOLTZMAN (1970) – *Nonlinear System Theory*. A Functional Analysis Approach, Prentice–Hall, Englewood Cliffs, New Jersey.

[6] A. HAMMERSTEIN (1930) – *Nichtlineare Integralgleichungen nebst Anwendungen*, « Acta Math. », *54*, 117–176.

[7] J. CROIN (1964) – *Fixed Points and Topological Degree in Nonlinear Analysis*. American Mathematicsl Society, Providence.

[8] E. B. LEE and L. MARKUS (1967) – *Foundations of Optimal Control Theory*. John Wiley and Sons, New York.

[9] M. A. KRASNOSEL'SKII (1964) – *Topological Methods in the Theory of Nonlinear Integral Equations*. Pergamon Press, Oxford.

[10] J. P. DAUER (1973) – *Perturbations of linear control systems*, « SIAM J. on Control », *9*, 393–400. Errata, *Ibid.*, *11*, 395.