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Maximal ideals in algebras of continuous C(S) valued functions

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Topologia. — *Maximal ideals in algebras of continuous* C (S) *valued functions*. Nota di WILLIAM HERY (*), presentata (**) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Siano S e T spazi completamente regolari, A l'algebra C (S , C) di tutte le funzioni continue su S a valori complessi con la topologia compatto-aperta, C (T , A) l'algebra di tutte le funzioni continue su T a valori in A, e $\mathcal{M}$(X) lo spazio di tutti gli ideali massimali di codimensione 1 nell'algebra X dotata di una certa topologia debole. Il principale risultato della Nota concerne l'esistenza di un isomorfismo di C (T , A) su C (T $\times \mathcal{M}$ (A) , F) quando S è localmente compatto e realcompatto.

Let S and T be completely regular spaces, A the topological algebra C (S , C) of all continuous complex valued functions on S with the compact-open topology, C (T , A) the algebra of all continuous A valued functions on T and $\mathcal{M}$ (X) the space of all maximal ideals of codimension 1 in an algebra X endowed with the weak topology generated by the family $\{\hat{x} : x \in X\}$, where $\hat{x}(m) = x + m$. If S is compact, C (S , C) becomes a B^* algebra and we can also consider the (Banach) algebra CB (T , A) of all bounded continuous functions from T into A. In [14] Yood showed that CB (T , A) is isometrically isomorphic to CB (T $\times \mathcal{M}$ (A) , C), and therefore $\mathcal{M}$ (CB (T , A)) is homeomorphic to the Stone-Cech compactification β (T $\times \mathcal{M}$(A)) (note that A and CB (T, A) are Banach algebras, so all maximal ideals are of codimension 1). For S realcompact [5] and locally compact, we obtain a similar isomorphism from C (T , A) onto C (T $\times \mathcal{M}$ (A) , C); this is then used to show that $\mathcal{M}$ (C (T , A)) is homeomorphic to the realcompactification ν (T $\times \mathcal{M}$ (A)). It is further shown that if S has nonmeasurable cardinal [5], $\mathcal{M}$ (C (T , A)) is also homeomorphic to (νT) $\times \mathcal{M}$ (A). These results are then used to prove two known topological results: if S is a finite discrete space and T is completely regular, then $\beta (T \times S) \cong (\beta T) \times S$ [6]; and if S is a locally compact, realcompact space with nonmeasurable cardinal, then $\nu (T \times S) \cong (\nu T) \times S$ [3]. Parallel results are obtained when A is the algebra C (S , F) of all continuous functions from an ultraregular space S into a nonarchimedean valued field F and T is ultraregular. (An ultraregular space is one in which there is a base for the topology consisting of sets which are simultaneously open and closed.) In that case, the Banaschewski (or nonarchimedean Stone-Cech) compactification [1 or 2] replaces the Stone-Cech compactification and the F-repletion [1] replaces the realcompactification. Either compactification will be denoted by β, both the realcompactification

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and F-repletion will be denoted by ν, and F will denote the underlying field (complex or nonarchimedean); alternate statements for the nonarchimedean case will be included in parentheses.

The properties of $C(S, F)$ for a completely regular (ultraregular) space S are well known ([5] and [1] are just two of several sources). Any f in $C(S, F)$ has a unique continuous extension νf in $C(\nu S, F)$ and $f \to \nu f$ is an isomorphism from $C(S, F)$ onto $C(\nu S, F)$. The F valued homomorphisms of $C(S, F)$ are precisely the evaluation maps $e_p(f) = \nu f(p)$, with $p \in \nu S$; all such homomorphisms are continuous when $C(S, F)$ has the compact-open topology. Identifying the F valued homomorphisms with their kernels (which must be closed maximal ideals of codimension 1), we consider $\mathcal{M}(A)$ to be a subset of the space A' of all continuous linear functionals on the topological vector space $A = C(S, F)$; the topology on $\mathcal{M}(A)$ generated by $\{\hat{f} : f \in A\}$ is then just the relative $\sigma(A', A)$ topology [13]. Furthermore, identifying the points of νS with $\mathcal{M}(A)$, the original topology on νS coincides with the relative $\sigma(A', A)$ topology. Thus, $\mathcal{M}(A)$ is homeomorphic to νS; in particular, if S is realcompact (F-replete) $\mathcal{M}(A)$ is homeomorphic to S.

The isomorphism we are interested in is given by

$$h: \begin{array}{c} C(T, A) \to C(T \times \mathcal{M}(A), F) \\ f \to \hat{f} \end{array} \quad \text{where} \quad \hat{f}(t, m) = f(t) + m .$$

The first problem encountered is the continuity of $\hat{f}$. With that in mind, we say that $\mathcal{M}(A)$ is locally equicontinuous if each point in $\mathcal{M}(A)$ has an equicontinuous neighborhood.

LEMMA 1. *Let* A *be any topological algebra for which* $\mathcal{M}(A)$ *is locally equicontinuous,* T *any completely regular (ultraregular) space and* $f \in C(T, A)$. *Then* $\hat{f}$ *is continuous.*

Proof. The continuity of $\hat{f}$ follows from the inequality

$$|\hat{f}(t, m) - \hat{f}(t_0, m_0)| \leq |f(t) + m - (f(t_0) + m)| + |f(t_0) + m - (f(t_0) + m_0)|,$$

the local equicontinuity of $\mathcal{M}(A)$ at $f(t_0)$, the continuity of f and the continuity of $m \to f(t_0) + m$.

The local equicontinuity of $\mathcal{M}(A)$ is not necessary to insure the continuity of each $\hat{f}$: if T is a discrete space, each $\hat{f}$ is continuous. For the restriction of $\hat{f}$ to each slice $\{t_0\} \times \mathcal{M}(A)$ is continuous by the definition of the topology on $\mathcal{M}(A)$, and the discreteness of T implies that $\{\{t_0\} \times \mathcal{M}(A) : t_0 \in T\}$ is an open partition of $T \times \mathcal{M}(A)$; thus $\hat{f}$ is continuous on $T \times \mathcal{M}(A)$.

LEMMA 2. *Let* S *be realcompact* (F-*replete*) *and* $A = C(S, F)$ *with the compact-open topology. Then* $\mathcal{M}(A)$ *is locally equicontinuous if and only if* S *is locally compact.*

Proof. The polar of a set $E \subset A$ is defined to be $E^0 = \{h \in A' : |h(f)| \leq 1 \ \forall f \in E\}$; a subset of A' is equicontinuous if and only if it is contained in the polar of a neighborhood of the origin in A [13]. A base at the origin for the compact open topology is the collection of sets $B_{K,r} = \{f \in A : |f(t)| \leq r$ for all $t \in K\}$, with K compact and $r \in (0, 1]$. Direct computation (using the complete regularity of S) shows that $(B_{K,r})^0 \cap \mathcal{M}(A) = K$. Thus the equicontinuous sets in $\mathcal{M}(A)$ are precisely the relatively compact sets, proving the lemma.

THEOREM 1. *Let* S *be locally compact and realcompact* (F-*replete*), $A = C(S, F)$ *with the compact open topology and* T *completely regular. Then* $h: C(T, A) \to C(T \times \mathcal{M}(A), F)$ *is an isomorphism.* $f \to \hat{f}$

Proof. Only the ontoness remains to be shown. Let $g \in C(T \times \mathcal{M}(A), F)$. For any $t_0 \in T$, $g(t_0, s) \in C(S, F)$; define $f(t_0) = g(t_0, \cdot)$. To show f is continuous at t_0, let $W = f(t_0) + B_{K,r}$ be a basic neighborhood of $f(t_0)$. For any $s_0 \in K$, the continuity of g implies the existence of neighborhoods $U_{s_0}(t_0)$ and $V(s_0)$ such that $t \in U_{s_0}(t_0)$ and $s \in V(s_0)$ imply $|g(t, s) - g(t_0, s_0)| \leq r/2$. Since K is compact, there exists a finite set $\{s_i\} \subset K$ such that $\{V(s_i)\}$ covers K. Let $U(t_0) = \cap \{U_{s_i}(t_0)\}$. Then if $t \in U(t_0)$ and $s \in K$, s must be in some $V(s_i)$ and

$$|f(t)(s) - f(t_0)(s)| \leq |g(t, s) - g(t_0, s_i)| + |g(t_0, s_i) - g(t_0, s)| \leq r.$$

Thus, $f(t) \in W$ and f is continuous; clearly $\hat{f} = g$.

The following Corollary to Theorem 1 is due to Yood [14]; the proof above is a modification of Yood's proof (which only applied to S compact and $F = C$).

COROLLARY 1. *If* A *is a* B* *algebra* (V* *Gelfand algebra with* F *locally compact*), *then* $CB(T, A)$ *is isometrically isomorphic to* $CB(T \times \mathcal{M}(A), F)$.

Proof. In the complex case, A is isometrically isomorphic to $C(\mathcal{M}(A), C)$ with the uniform norm; clearly f is bonded if and only if $\hat{f}$ is, and $\|f\| = \|\hat{f}\|$. The same argument applies in the nonarchimedean case, noting that A is isometrically isomorphic to $C(\mathcal{M}(A), F)$ if $\mathcal{M}(A)$ is compact [12, p. 165], which is the case if F is locally compact [12, p. 124]. An example in [8] shows that the local compactness of F cannot be dropped, even if T consists of only one point.

COROLLARY 2. *Let* A *and* T *be as in Theorem 1. Then* $\mathcal{M}(C(T, A))$ *is homeomorphic to* $\nu(T \times \mathcal{M}(A))$. *In particular, if* T *is also realcompact* (F-*replete*), $\mathcal{M}(C(T, A))$ *is homeomorphic to* $T \times \mathcal{M}(A)$.

The conclusion $\mathcal{M}(C(T, A)) \cong T \times \mathcal{M}(A)$ has been obtained by Hausner [7] when T is compact and A is a Banach algebra, by Mallios [10] when T is compact and A is a locally multiplicatively convex topological algebra whose completion is a Q algebra and by Dietrich [4] when T is a completely regular k-space and A is a complete locally convex topological algebra with $\mathcal{M}(A)$ locally equicontinuous; the latter two used the space of all closed maximal ideals of codimension 1 in lieu of $\mathcal{M}(X)$ as used here and give

C (T , A) the compact-open topology. The results here neither supercede nor are superceded by the above results. When T is a completely regular realcompact k-space and S in locally compact and realcompact, C (S , F) is complete (see below) and both Corollary 1 and Dietrich's result both apply. Since the elements of $\mathscr{M}$ (A) are all closed, we see that the elements of $\mathscr{M}$ (C (T , A)) are also all closed; i.e., all F valued homomorphisms of C (T , A) are continuous.

COROLLARY 3. *Let* A *and* T *be as in Corollary 1. Then* $\mathscr{M}$ (C (T, A)) *is homeomorphic to* $\beta(\mathrm{T}\times\mathscr{M}(\mathrm{A}))$.

Proof. Since F is locally compact, $\mathrm{CB}(\mathrm{T}\times\mathscr{M}(\mathrm{A}),\mathrm{F}) = \mathrm{C}^*(\mathrm{T}\times\mathscr{M}(\mathrm{A}),\mathrm{F})$ (the algebra of continuous functions with relatively compact range). In both the complex and nonarchimedean [1, Theorem 7] cases, the maximal ideal space is then known to be $\beta(\mathrm{T}\times\mathscr{M}(\mathrm{A}))$.

Another representation of $\mathscr{M}$ (C (T , A)) is available when A is also realcompact (F-replete), which will be true when S is locally compact and S and F have nonmeasurable cardinal. For then S is a k-space [9, p. 231] and C (S , F) is complete in the compact-open topology [*ibid.*]. (This applies in both the complex and nonarchimedean cases, since the only property of F used is that it is a uniform space). Shirota's Theorem [5, p. 232] then shows that A is realcompact. The cardinality restriction is not too severe, since a measurable cardinal must be an inaccessable cardinal, and the nonexistence of inaccessable cardinals is consistent with the Zermelo-Frankel axioms of set theory with the axiom of choice; it is therefore not possible to prove the existence of measurable cardinals within that axiom system.

COROLLARY 4. *Let* T, S *and* A *be as in Theorem 1 and* A *realcompact* (F-*replete*). *Then* $\mathscr{M}$ (C (T , A)) *is homeomorphic to* $(\nu\mathrm{T})\times\mathscr{M}(\mathrm{A})$.

Proof. By the realcompactness (F-repleteness) of A, each $f \in \mathrm{C}(\mathrm{T},\mathrm{A})$ has a unique extension $\nu f \in \mathrm{C}(\nu\mathrm{T},\mathrm{A})$ and $f \to \nu f$ is an isomorphism. Applying Corollary 2 and using the realcompactness of νT and S yields the desired result.

We next use these results to prove two known topological results. (These results have been published for the complex case. I am not aware of anything in print regarding the nonarchimedean case, but it seems that proofs analogous to those published for the complex case would be possible.)

PROPOSITION 1 (Glicksberg). *Let* S *be a finite discrete space and* T *completely regular* (*ultraregular*). *Then* $\beta(\mathrm{T}\times\mathrm{S}) \cong (\beta\mathrm{T})\times\mathrm{S}$.

Proof. Let F be the complex numbers (any locally compact nonarchimedean valued field, such as the 2-adic numbers) and A = C (S , F) with the uniform norm. Since $\mathscr{M}(\mathrm{A}) \cong \mathrm{S}$, it sollows from Corollary 3 that $\mathscr{M}(\mathrm{CB}(\mathrm{T},\mathrm{A})) \cong \beta(\mathrm{T}\times\mathrm{S})$. Using the local compactness of A, we obtain

$$\mathrm{CB}(\mathrm{T},\mathrm{A}) = \mathrm{C}^*(\mathrm{T},\mathrm{A}) \cong \mathrm{C}(\beta\mathrm{T},\mathrm{A}) = \mathrm{CB}(\beta\mathrm{T},\mathrm{A}).$$

Applying Corollary 3 again, we obtain $\mathscr{M}(\mathrm{CB}(\mathrm{T},\mathrm{A})) \cong (\beta\mathrm{T})\times\mathrm{S}$ proving the corollary.

PROPOSITION 2 (Comfort-Negrepontis). *Let* S *be a locally compact and realcompact* (*locally compact, ultraregular and* F-*replete*) *space with nonmeasurable cardinal, and* T *completely regular* (*ultraregular*). *Then* $\nu(T \times S) \cong (\nu T) \times S$.

Proof. Let $A = C(S, F)$. From the remarks preceding Corollary 4, we see that A is realcompact (F-replete); the proposition then follows from Corollaries 2 and 4.

In [3], Comfort explores the consequences of this proposition in relation to the question of when $\nu(T \times S)$ is homeomorphic to $(\nu T) \times (\nu S)$.

We conclude by noting that Theorem 1, Corollaries 2 and 4, and Proposition 2 are also true if T is discrete and S is a k-space (instead of a locally compact space) (see the remarks following Lemma 1). Since a discrete space is realcompact if and only if it has nonmeasurable cardinal, the conclusion of Proposition 2 in this case is trivial when T has unmeasurable cardinal; but if T has measurable cardinal and S is a realcompact k-space with unmeasurable cardinal, $\nu(T \times S) \cong (\nu T) \times S = (\nu T) \times (\nu S)$ is of interest. This condition can then be added to the list of sufficient conditions developed by Comfort in [3] for $\nu(T \times S)$ to be homeomorphic to $(\nu T) \times (\nu S)$.

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