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Some special Ricci identities

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Geometria differenziale. — *Some special Ricci identities.* Nota di H. D. PANDE e S. B. MISRA, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Estensione di identità del Ricci (nel caso di spazi riemanniani) agli spazi di Finsler.

1. INTRODUCTION

In a Finsler space F_n [1] (1) we have two types of covariant derivatives of a vector field $X^i(x, \dot{x})$ with respect to x^k, given by

$$X^i|_k = F\dot{\partial}_k X^i + X^m A^i_{mk} \text{ (2)} \tag{1.1}$$

and

$$X^i_{|k} = \partial_k X^i - \dot{\partial}_m X^i G^m_k + X^m \Gamma^{*i}_{mk} \tag{1.2}$$

where

$$A^i_{jk}(x, \dot{x}) \overset{\text{def}}{=} F c^i_{jk}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} F g^{ih} \dot{\partial}_k g_{jh} \tag{1.3}$$

are the components of a symmetric tensor and $\Gamma^{*i}_{hk}(x, \dot{x})$ are Cartan's connection coefficients.

The projective covariant derivative [3] of a vector fields $X^i(x, \dot{x})$ with respect to x^k is given by

$$X^i \|_k = F\dot{\partial}_k X^i + X^m \Pi^i_{mk} \tag{1.4}$$

where $\Pi^i_{mk}(x, \dot{x})$ are projective connection parameters, defined by

$$\Pi^i_{mk} = G^i_{mk} - \frac{1}{n+1} (\delta^i_k G^r_{rm} + \delta^i_m G^r_{rk} + \dot{x}^i G^r_{rkm}) . \tag{1.5}$$

The functions $G^i_{jk}(x, \dot{x})$ are Berwald's connection coefficients (Rund [1]). The commutation formula [3] involving both (1.4) and (1.2) processes is given by

$$X^i \|_{h|k} - X^i_{|k} \|_h = -M^i_{jkh} X^j + X^i \|_j \Pi^j_{hk|m} l^m + X^i_{|j} \Pi^j_{hk} \tag{1.6}$$

where

$$M^i_{jkh}(x, \dot{x}) \overset{\text{def}}{=} F\dot{\partial}_h \Gamma^{*i}_{jk} + \Pi^i_{jm} \Pi^m_{hk|r} l^r - \Pi^i_{jh|k} \tag{1.7}$$

and

$$l^i(x, \dot{x}) \overset{\text{def}}{=} \frac{\dot{x}^i}{F(x, \dot{x})} . \tag{1.8}$$

(*) Nella seduta dell'8 febbraio 1975.

(1) Numbers in square brackets refer to the references at the end of the paper.

(2) $\partial_k \equiv \partial/\partial x^k$ and $\dot{\partial}_k \equiv \partial/\partial \dot{x}^k$.

2. RICCI IDENTITIES

THEOREM (2.1). *The Ricci identity for a covariant tensor of order two is given by*

(2.1) $$A_{ij}\|_{h|k} - A_{ij|k}\|_h = A_{pj} M^p_{ikh} + A_{ip} M^p_{jkh} + A_{ij}\|_p \Pi^p_{hk|m} l^m + A_{ij|m} \Pi^m_{hk}.$$

Proof. Let $X^i(x, \dot{x})$ be an arbitrary contravariant vector field such that its inner product with the tensor $A_{ij}(x, \dot{x})$ is given by

(2.2) $$T_i(x, \dot{x}) \overset{\text{def}}{=} A_{ij}(x, \dot{x}) X^j.$$

We know that

(2.3) $$T_i\|_{h|k} - T_{i|k}\|_h = T_j M^j_{ikh} + T_i\|_j \Pi^j_{hk|m} l^m + T_{i|m} \Pi^m_{hk}.$$

With the help of (2.2), equation (2.3) takes the form

(2.4) $$X^j [A_{ij}\|_{h|k} - A_{ij|k}\|_h] + A_{ij} [X^j\|_{h|k} - X^j_{|k}\|_h] = M^j_{ikh} A_{jp} X^p + \\ + A_{ij}\|_p \Pi^p_{hk|m} l^m X^j + A_{ij} X^j\|_p \Pi^p_{hk} l^m + A_{ij|m} \Pi^m_{hk} X^j + \\ + A_{ij} X^j_{|m} \Pi^m_{hk}.$$

Taking into account (1.6) and rearranging the terms in (2.4) we have

(2.5) $$X^j [A_{ij}\|_{h|k} - A_{ij|k}\|_h - A_{ip} M^p_{jkh} - A_{pj} M^p_{ikh} - A_{ij}\|_p \Pi^p_{hk|m} l^m - \\ - A_{ij|m} \Pi^m_{hk}] = 0.$$

Since $X^j(x, \dot{x})$ is an arbitrary vector, (2.1) follows from (2.5).

THEOREM (2.2). *The Ricci identity for a contravariant tensor of order two is given by*

(2.6) $$A^{ij}\|_{h|k} - A^{ij}_{|k}\|_h = -A^{ij}\|_p \Pi^p_{hk|m} l^m - A^{ij}_{|p} \Pi^p_{hk} - A^{ip} M^j_{ikh} - A^{pj} M^i_{pkh}.$$

Proof. Let $X_i(x, \dot{x})$ be an arbitrary covariant vector field such that its inner product with the tensor $A^{ij}(x, \dot{x})$ is given by

(2.7) $$T^i(x, \dot{x}) = A^{ij}(x, \dot{x}) X_j.$$

We know that

(2.8) $$T^i\|_{h|k} - T^i_{|k}\|_h = -M^i_{jkh} T^j + T^i\|_j \Pi^j_{hk|m} l^m + T^i_{|j} \Pi^j_{hk}$$

With the help of (2.7), equation (2.8) takes the following form

(2.9) $$X_j [A^{ij}\|_{h|k} - A^{ij}_{|k}\|_h] + A^{ij} [X_j\|_{h|k} - X_{j|k}\|_h] = -M^i_{jkh} A^{jp} X_p + \\ + A^{ij}\|_p \Pi^p_{hk|m} l^m X_j + A^{ij} X_j\|_p \Pi^p_{hk|m} l^m + A^{ij}_{|p} \Pi^p_{hk} X_j + A^{ij} X_{j|p} \Pi^p_{hk}.$$

Using equations (1.8) and (2.9), we get

$$X_j \,[A^{ij}\,\|_{h|k} - A^{ij}_{|k}\,\|_h + A^{ip}\,M^j_{pkh} + A^{pj}\,M^i_{pkh} + A^{ij}\,\|_p\,\Pi^p_{hk|m}\,l^m + \\ + A^{ij}_{|p}\,\Pi^p_{hk}] = 0 . \tag{2.10}$$

Since $X_j(x, \dot{x})$ is an arbitrary vector, (2.6) follows from (2.10).

THEOREM (2.3). *The Ricci identity for a covariant tensor of arbitrary rank q is given by*

$$A_{i_1,i_2,\cdots,i_q|k}\,\|_{h|k} - A_{i_1,i_2,\cdots,i_q|k}\,\|_h = A_{i_1,i_2,\cdots,i_q}\,\|_p\,\Pi^p_{hk|s}\,l^s + \\ + A_{i_1,i_2,\cdots,i_q|p}\,\Pi^p_{hk} + \sum_{\beta=1}^{q} A_{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_q}\,M^s_{i_\beta kh} . \tag{2.11}$$

Proof. Let us assume that the theorem holds for a covariant tensor of order $m\,(m < q)$. Thus, we have

$$T_{i_1,i_2,\cdots,i_m}\,\|_{h|k} - T_{i_1,i_2,\cdots,i_m|k}\,\|_h = T_{i_1,i_2,\cdots,i_m|p}\,\Pi^p_{hk} + \\ + T_{i_1,i_2,\cdots,i_m}\,\|_p\,\Pi^p_{hk|s}\,l^s + \sum_{\beta=1}^{m} T_{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_m}\,M^s_{i_\beta kh} . \tag{2.12}$$

Let $A_{i_1,i_2,\cdots,i_m,j}(x, \dot{x})$ be a $(m+1)$ order covariant tensor and $X^j(x, \dot{x})$ as before is an arbitrary contravariant vector. The inner product of $X^j(x, \dot{x})$ and $A_{i_1,i_2,\cdots,j_m,j}(x, \dot{x})$ is given by

$$T_{i_1,i_2,\cdots,i_m}(x, \dot{x}) \overset{\text{def}}{=} A_{i_1,i_2,\cdots,i_m,j}(x, \dot{x})\,X^j. \tag{2.13}$$

Substituting the value of $T_{i_1,i_2,\cdots,i_m}(x, \dot{x})$ from (2.13) in (2.12), we have

$$X^j\,[A_{i_1,i_2,\cdots,i_m,j}\|_{h|k} - A_{i_1,i_2,\cdots,i_m|k}\,\|_h] + A_{i_1,i_2,\cdots,i_m,j}\,[X^j\,\|_{h|k} - X^j_{|k}\,\|_h] = \\ = A_{i_1,i_2,\cdots,i_m,j|p}\,\Pi^p_{hk}\,X^j + A_{i_1,i_2,\cdots,i_m,j}\,X^j_{|p}\,\Pi^p_{hk} + \\ + A_{i_1,i_2,\cdots,i_m,j}\,\|_p\,\Pi^p_{hk|s}\,l^s\,X^j + A_{i_1,i_2,\cdots,i_m,j}\,X^j\,\|_p\,\Pi^p_{hk|s}\,l^s + \\ + \sum_{\beta=1}^{m} A_{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_m,j}\,M^s_{i_\beta kh}\,X^j . \tag{2.14}$$

Taking into account (1.8) and rearranging the terms in (2.14) we obtain

$$X^j\,[A_{i_1,i_2,\cdots,i_m,j}\,\|_{h|k} - A_{i_1,i_2,\cdots,i_m,j|k}\,\|_h - A_{i_1,i_2,\cdots,i_m,p}\,M^p_{jkh} - \\ - A_{i_1,i_2,\cdots,i_m,j|p}\,\Pi^p_{yk} - A_{i_2,i_2,\cdots,i_m,j}\,\|_p\,\Pi^p_{hk|s}\,l^s - \\ - \sum_{\beta=1}^{m} A_{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_m,j}\,M^s_{i_\beta kh}] = 0 . \tag{2.15}$$

Replacing the index j by i_{m+1} in (2.15) and taking into account the fact that $X^j(x, \dot{x})$ is an arbitrary vector, we obtain

$$(2.16)\qquad A_{i_1,i_2,\cdots,i_{m+1}}\|_{h|k} - A_{i_1,i_2,\cdots,i_{m+1}|k}\|_h = A_{i_1,i_2,\cdots,i_{m+1}|p}\,\Pi^p_{hk} + A_{i_1,i_2,\cdots,i_{m+1}}\|_p\,\Pi^p_{hk|s}\,l^s + \sum_{\beta=1}^{m+1} A_{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_{m+1}}\,M^s_{i_\beta kh}\,.$$

Thus the formula is true also for a covariant tensor of order $(m+1)$. It has already been established that the formula is true for a covariant tensor of order two. Hence it is true for a covariant tensor of any, order say q.

THEOREM (2.4). *The Ricci identity for a contravariant tensor of an arbitrary rank q is given by*

$$(2.17)\qquad A^{i_1,i_2,\cdots,i_q}\|_{h|k} - A^{i_1,i_2,\cdots,i_q}{}_{|k}\|_h = A^{i_1,i_2,\cdots,i_q}{}_{|p}\,\Pi^p_{hk} + A^{i_1,i_2,\cdots,i_q}\|_p\,\Pi^p_{hk|s}\,l^s - \sum_{\beta=1}^{q} A^{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_q}\,M^{i_\beta}_{skh}\,.$$

Proof. Let us suppose that the theorem holds for a contravariant tensor of order m $(m < q)$. Thus, we have

$$(2.18)\qquad T^{i_1,i_2,\cdots,i_m}\|_{h|k} - T^{i_1,i_2,\cdots,i_m}{}_{|k}\|_h = T^{i_1,i_2,\cdots,i_m}{}_{|p}\,\Pi^p_{hk} + T^{i_1,i_2,\cdots,i_m}\|_p\,\Pi^p_{hk|s}\,l^s - \sum_{\beta=1}^{m} T^{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_m}\,M^{i_\beta}_{skh}\,.$$

Let $A^{i_1,i_2,\cdots,i_m,j}(x, \dot{x})$ be a $(m+1)$ order contravariant tensor and $X_i(x, \dot{x})$ be an arbitrary covariant vector field. The inner product of $X_i(x, \dot{x})$ and $A^{i_1,i_2,\cdots,i_m,j}(x, \dot{x})$ is given by

$$(2.19)\qquad T^{i_1,i_2,\cdots,i_m}(x, \dot{x}) \overset{\text{def}}{=} A^{i_1,i_2,\cdots,i_m,j}(x, \dot{x})\,X_j\,.$$

Substituting the value of $T^{i_1,i_2,\cdots,i_m}(x, \dot{x})$ from (2.19) in (2.18) we have

$$(2.20)\qquad X_j[A^{i_1,i_2,\cdots,i_m,j}\|_{h|k} - A^{i_1,i_2,\cdots,i_m,j}{}_{|k}\|_h] + A^{i_1,i_2,\cdots,i_m,j}[X_j\|_{h|k} - X_{j|k}\|_h] = A^{i_1,i_2,\cdots,i_m,j}{}_{|p}\,\Pi^p_{hk}\,X_j + A^{i_1,i_2,\cdots,i_m,j}\,X_{j|p}\,\Pi^p_{hk} + A^{i_1,i_2,\cdots,i_m,j}\|_p\,\Pi^p_{hk|s}\,l^s\,X_j + A^{i_1,i_2,\cdots,i_m,j}\,X_j\|_p\,\Pi^p_{hk|s}\,l^s - \sum_{\beta=1}^{m} A^{i_1,i_2,\cdots,i_{\beta-1},s,\beta+1,\cdots,i_m,j}\,M^{i_\beta}_{skh}\,X_j\,.$$

With the help of equations (1.8) and (2.20), we obtain

$$(2.21)\qquad X_j\Big[A^{i_1,i_2,\cdots,i_m,j}\|_{h|k} - A^{i_1,i_2,\cdots,i_m,j}{}_{|k}\|_h + A^{i_1,i_2,\cdots,i_m,p}\,M^j_{pkh} - A^{i_1,i_2,\cdots,i_m,j}{}_{|p}\,\Pi^p_{hk} - A^{i_1,i_2,\cdots,i_m,j}\|_p\,\Pi^p_{hk|s}\,l^s + \sum_{\beta=1}^{m} A^{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_m,j}\,M^{i_\beta}_{skh}\Big] = 0\,.$$

Replacing the index j by i_{m+1} in (2.21) and taking into account the fact that $X_j(x, \dot{x})$ is an arbitrary covariant vector, we obtain

$$(2.22) \qquad A^{i_1,i_2,\cdots,i_{m+1}}\|_{h|k} - A^{i_1,i_2,\cdots,i_{m+1}}{}_{|k}\|_h = A^{i_1,i_2,\cdots,i_{m+1}}{}_{|p}\, \Pi^p_{hk} +$$

$$+ A^{i_1,i_2,\cdots,i_{m+1}}\|_p\, \Pi^p_{hk|s}\, l^s - \sum_{\beta=1}^{m+1} A^{i_1,i_2,\cdots,i_{\beta-1},s,i_{\beta+1},\cdots,i_{m+1}}\, M^{i_\beta}_{skh}$$

(2.22) shows that the theorem is true for a contravariant tensor of order $(m+1)$. It has already been shown that the formula is true for a contravariant tensor of order two. Hence according to induction principle the formula is true for a tensor of an arbitrary rank say q.

THEOREM (2.5). *The Ricci identity for a mixed tensor (contravariant p and covariant q) is given by*

$$(2.23) \qquad A^{i_1,i_2,\cdots,i_p}_{j_1,j_2,\cdots,j_q}\|_{h|k} - A^{i_1,i_2,\cdots,i_p}_{j_1,i_2,\cdots,j_q|k}\|_h = A^{i_1,i_2,\cdots,i_p}_{j_1,j_2,\cdots,j_q|s}\, \Pi^s_{hk} +$$

$$+ A^{i_1,i_2,\cdots,i_p}_{j_1,j_2,\cdots,j_q}\|_m\, \Pi^m_{hk|s}\, l^s + \sum_{\beta=1}^{q} A^{i_1,i_2,\cdots,i_p}_{j_1,j_2,\cdots,j_{\beta-1},s,j_{\beta+1},\cdots,j_q}\, M^s_{j_\beta kh} -$$

$$- \sum_{\alpha=1}^{p} A^{i_1,i_2,\cdots,\alpha-1,s,i_{\alpha+1},\cdots,i_p}_{j_1,j_2,\cdots,j_q}\, M^{i_\alpha}_{skh} .$$

Proof. The proof of the Theorem (2.5) follows the pattern of the proofs of the Theorems (2.3) and (2.4).

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