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A. Kumar

- On special projective tensor fields

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Geometria differenziale. — *On special projective tensor fields.* Nota di A. KUMAR, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Estensione agli spazi di Finsler delle identità del Bianchi e del Veblen per campi tensoriali di curvatura speciali.

1. INTRODUCTION

Let F_n [1] (1) be an n-dimensional Finsler space equipped with $2n$ line elements $(x^i, \dot{x}^i)$ and a fundamental function $F(x^i, \dot{x}^i)$ positively homogeneous of degree two in its directional arguments. The fundamental metric tensor $g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})$ is symmetric in its indices i and j. Let $X^i(x, \dot{x})$ be a contravariant vector field depending on both the positional and directional coordinates. The covariant derivative of $X^i(x, \dot{x})$ with respect to x^k in the sense of Berwald is given by

$$X^i_{(k)} = \partial_k X^i - (\dot{\partial}_h X^i) G^h_k + X^h G^i_{hk}, \tag{1.1}$$

where

$$G^i(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{4} g^{ih} \{2 \partial_{(j} g_{k)h} - \partial_h g_{jk}\} \dot{x}^j \dot{x}^k \tag{1.2}$$

are Berwald's connection coefficients, positively homogeneous of degree two in $\dot{x}^i$, and satisfying the following identities:

$$G^i_{hkr} \dot{x}^h = 0 \quad , \quad G^i_{hk} \dot{x}^h = G^i_k \quad \text{and} \quad G^i_k \dot{x}^k = 2 G^i . \tag{1.3}$$

The Berwald covariant derivatives of $F(x, \dot{x})$ and $g_{ih}(x, \dot{x})$ vanish i.e.

$$F_{(h)} = 0 \quad \text{and} \quad g_{ih(k)} = 0 . \tag{1.4}$$

Misra [3] has defined the projective covariant derivative of $X^i(x, \dot{x})$ as follows

$$X^i_{((k))} = \partial_k X^i - (\dot{\partial}_h X^i) \Pi^h_{kr} \dot{x}^r + X^h \Pi^i_{hk}, \tag{1.5}$$

where

$$\Pi^i_{hk}(x, \dot{x}) \overset{\text{def}}{=} G^i_{hk} - \frac{1}{(n+1)} (2 \delta^i_{(h} G^r_{k)r} + \dot{x}^i G^r_{rkh}) \tag{1.6}$$

are called the projective connection coefficients. The entities $\Pi^i_{hk}(x, \dot{x})$ are

(*) Nella seduta dell'8 febbraio 1975.

(1) Numbers in brackets refer to the references at the end of the paper.

(2) $2 A_{(hk)} = A_{hk} + A_{kh}$, $2 A_{[hk]} = A_{hk} - A_{kh}$.

positively homogeneous of degree zero in $\dot{x}^i$ and satisfy the following relations:

$$\Pi^i_{hkr}\,\dot{x}^h = 0 \quad , \quad \Pi^i_{hk}\,\dot{x}^k = \Pi^i_k \quad , \quad \Pi^i_j\,\dot{x}^j = 2\,\Pi^i , \tag{1.7}$$
$$\dot{\partial}_j\,\Pi^i_{hk} = \Pi^i_{jhk} \qquad \text{and} \qquad \dot{\partial}_h\,\Pi^i_k = \Pi^i_{hk} .$$

The projective curvature tensor field $H^i_{jhk}(x,\dot{x})$ arising from Berwald's covariant derivative (1.1) is given by

$$H^i_{hjk}(x,\dot{x}) \overset{\text{def}}{=} \frac{2}{3}\,\dot{\partial}_h\,\dot{\partial}_{[j}\,H^i_{k]} = 2\,\{\partial_{[k}\,G^i_{j]h} + G^r_{h[j}\,G^i_{k]r} + G^i_{rh[k}\,G^r_{j]}\} . \tag{1.8}$$

The tensor fields $H^i_{jkh}(x,\dot{x})$ and $H^i_{kh}(x,\dot{x})$ satisfy the following identities:

(1.9) *a*) $H^i_{jhk}\,\dot{x}^j = H^i_{hk}$ *b*) $H^i_{hk}\,\dot{x}^h = H^i_k$

c) $H^i_k\,\dot{x}^k = 0$ *d*) $H^i_{jkh} = -\,H^i_{jhk}$

(1.10) *a*) $H^i_{j[hk(l)]} + H^m_{[hk}\,G^i_{l]mj}$ [4] *b*) $H^r_{[ik(l)]} = 0$ *c*) $H^i_{[ljh]} = 0$

and

(1.11) *a*) $H^i_{hji} = H_{hj}$ *b*) $H^i_{ijk} = 2\,H_{[jk]}$

c) $H^i_{ji} = H_j$ *d*) $H^i_i = (n-1)\,H$.

The projective entities Q^i_{jhk} satisfies the following relations:

(1.12) *a*) $Q^i_{[jkh]} = 0$, *b*) $Q^i_{j(kh)} = 0$, *c*) $Q^i_{[jk(s)]} = 0$,

(1.13) *a*) $Q^i_{jhk}\,\dot{x}^j = Q^i_{hk}$, *b*) $Q^i_{jk}\,\dot{x}^j = Q^i_k$, *c*) $Q^i_k\,\dot{x}^k = 0$,

d) $Q^i_{jkh} = -\,Q^i_{jhk}$

and

$$Q^i_{ji} = Q_j \quad , \quad Q^i_{ikj} = 2\,Q_{[jk]} \quad \text{and} \quad Q^i_{jki} = Q_{jk} . \tag{1.14}$$

2. SPECIAL PROJECTIVE TENSOR FIELDS

Let us consider the linear combination of the projective entity $Q^i_j(x,\dot{x})$ and of the deviation tensor field $H^i_j(x,\dot{x})$ given by:

$$M^i_j(x,\dot{x}) \overset{\text{def}}{=} F(x,\dot{x})\,H^i_j + P(x,\dot{x})\,Q^i_j , \tag{2.1}$$

where $P(x,\dot{x})$ is a scalar homogeneous function of degree one in $\dot{x}^i$. We

(3) $\dot{\partial}_i \equiv \partial/\partial\dot{x}^i$, $\partial_h \equiv \partial/\partial x^h$ and $\dot{\partial}^2_{lh} \equiv \partial^2/\partial\dot{x}^l\,\partial\dot{x}^h$.

(4) $A_{[ijk]} = {}^1/_3\,[A_{ijk} + A_{jki} + A_{kij}]$.

call $M^i_j(x, \dot{x})$ the special projective tensor field. Contracting equation (2.1) with respect to the indices i and j and using (1.11), we obtain

$$M^i_i(x, \dot{x}) = (n-1)\, FH + PQ^i_i \,. \tag{2.2}$$

Transvecting (2.1) by $\dot{x}^j$ and using equations (1.19 c) and (1.13 c), we get

$$M^i_j \dot{x}^j = 0 \,. \tag{2.3}$$

On differentiating partially with respect to $\dot{x}^h$, equation (2.3) yields

$$(\dot{\partial}_h M^i_j)\, \dot{x}^j + M^i_h = 0 \,. \tag{2.4}$$

We now derive further special projective tensor fields from (2.1) by applying the same method as adopted for the projective curvature tensor fields and projective entities. We write them as

$$a)\ M^i_{hj}(x, \dot{x}) \overset{\text{def}}{=} \frac{2}{3}\, \dot{\partial}_{[h} M^i_{j]} \qquad b)\ M^i_{lhj}(x, \dot{x}) \overset{\text{def}}{=} \dot{\partial}_l M^i_{hj} = \frac{2}{3}\, \dot{\partial}^2_{l[h} M^i_{j]} \,. \tag{2.5}$$

With the help of the above equations the tensor fields $M^i_{hj}(x, \dot{x})$ and $M^i_{lhj}(x, \dot{x})$ can be expressed in the forms

$$M^i_{hj}(x, \dot{x}) = FH^i_{hj} + PQ^i_{hj} + \frac{2}{3}\, [\dot{\partial}_{[h} PQ^i_{j]} + \dot{\partial}_{[h} FH^i_{j]}] \tag{2.6}$$

and

$$\begin{aligned} M^i_{lhj}(x, \dot{x}) = {} & FH^i_{lhj} + PQ^i_{lhj} + (\dot{\partial}_l F)\, H^i_{hj} + (\dot{\partial}_l P)\, Q^i_{hj} + \\ & + \frac{2}{3}\, [\dot{\partial}^2_{l[h} PQ^i_{j]} + \dot{\partial}_{[h} P \dot{\partial}_{\langle l \rangle} Q^i_{j]} \ ^{(5)} + 2 g_{l[h} H^i_{j]} + \dot{\partial}_{[h} F \dot{\partial}_{\langle l \rangle} H^i_{j]} \,. \end{aligned} \tag{2.7}$$

Transvecting (2.6) by $\dot{x}^h$ and using equations (1.9 b), (2.1) and the homogeneity properties of the scalar functions $P(x, \dot{x})$ and $F(x, \dot{x})$, we get

$$M^i_{hj} \dot{x}^h = \frac{4}{3}\, M^i_j \,. \tag{2.8}$$

Similarly on transvecting (2.7) by $\dot{x}^{'s}$ and using equations (1.9), (1.13), (2.6) and the homogeneity properties of $P(x, \dot{x})$ and $F(x, \dot{x})$, we obtain

$$M^i_{lhj} \dot{x}^l = 2\, M^i_{hj} \qquad \text{and} \qquad M^i_{lhj} \dot{x}^l \dot{x}^h = \frac{\theta}{3}\, M^i_j \,. \tag{2.9}$$

Contracting (2.6) and (2.7) with respect to the indices i, j and using equations (1.11) and (1.14), we have

$$\begin{aligned} M^i_{hi} = PQ_h + FH_h + \frac{1}{3}\, [& (n-1)\, (\dot{\partial}_h F)\, H + (\dot{\partial}_h P)\, Q^i_i - \\ & - (\dot{\partial}_i PQ^i_h + \dot{\partial}_i FH^i_h)] \end{aligned} \tag{2.10}$$

(5) The indices in brackets $\langle\,\rangle$ are free from symmetric and skew symmetric parts.

and

(2.11)
$$M^i_{lhi} = PQ_{lh} + FH_{lh} + \dot\partial_l PQ_h + \dot\partial_l FH_h +$$
$$+ \frac{1}{3} [\dot\partial^2_{lh} PQ^i_i + \dot\partial^2_{li} PQ^i_h + \dot\partial_h P \dot\partial_l Q^i_i - \dot\partial_i P \dot\partial_l Q^i_h$$
$$+ 2((n-1) g_{lh} H - g_{li} H^i_h) + (n-1) \dot\partial_h F \dot\partial_l H - \dot\partial_i F \dot\partial_l H^i_h]$$

respectively.

We have the following theorems depending on the special projective tensor fields.

THEOREM (2.1). *The Bianchi identities for the special projective tensor fields* $M^i_{hj}(x, \dot x)$ *and* $M^i_{lhj}(x, \dot x)$ *are given by*

(2.12)
$$M^i_{[hj(s)]} = P_{[(s)} Q^i_{hj]} + \frac{1}{3} [\dot\partial_{[h} P_{(s)} Q^i_{j]} - \dot\partial_{[j} P_{(s)} Q^i_{h]} +$$
$$+ \dot\partial_{[h} PQ^i_{j(s)]} - \dot\partial_{[j} PQ^i_{h(s)]} + \dot\partial_{[h} FH^i_{j(s)]} - \dot\partial_{[j} FH^i_{h(s)]}]$$

and

(2.13)
$$M^i_{l[hj(s)]} = Q^i_{l[hj} P_{(s)]} + PQ^i_{l[hj(s)]} - FH^m_{[hj} G^i_{s]lm} +$$
$$+ \dot\partial_l P_{[(s)} Q^i_{hj]} + \frac{1}{3} [(\dot\partial^2_{l[h} P_{(s)} + \dot\partial_{\langle m\rangle} PG^m_{\langle l\rangle hs}) Q^i_{j]} -$$
$$+ (\dot\partial^2_{l[j} P_{(s)} + \dot\partial_{\langle m\rangle} PG^m_{\langle l\rangle js}) Q^i_{h]} + (\dot\partial_{[h} P_{(s)} \dot\partial_{\langle l\rangle} Q^i_{j]} - \dot\partial_{[j} P_{(s)} \dot\partial_{\langle l\rangle} Q^i_{h]}) +$$
$$+ (\dot\partial_l Q^i_{[j(s)} + Q^i_m G^m_{l[js} - Q^m_{[j} G^i_{\langle ml\rangle s}) \dot\partial_{h]} P -$$
$$- (\dot\partial_l Q^i_{[h(s)} + Q^i_m G^m_{l[hs} - Q^m_{[h} G^i_{\langle ml\rangle s}) \dot\partial_{j]} P + 2 (g_{l[h} H^i_{j(s)]} - g_{l[j} H^i_{h(s)]}) +$$
$$+ (\dot\partial_l H^i_{[j(s)} + H^i_m G^m_{l[js} - H^m_{[j} G^i_{\langle ml\rangle s}) \dot\partial_{h]} F -$$
$$- (\dot\partial_l H^i_{[h(s)} + H^i_m G^m_{l[hs} - H^m_{[h} G^i_{\langle ml\rangle s}) \dot\partial_{j]} F] .$$

Proof. Differentiating (2.6) and (2.7) covariantly with respect to x^s in the sense of Berwald and using equation (1.4), we get respectively

(2.14)
$$M^i_{hj(s)} = FH^i_{hj(s)} + P_{(s)} Q^i_{hj} + PQ^i_{hj(s)} +$$
$$+ \frac{2}{3} [\dot\partial_{[h} P_{(s)} Q^i_{j]} + \dot\partial_{[h} PQ^i_{j](s)} + \dot\partial_{[h} FH^i_{j](s)}]$$

and

(2.15)
$$M^i_{lhj(s)} = FH^i_{lhj(s)} + P_{(s)} Q^i_{lhj(s)} + PQ^i_{lhj(s)} + \dot\partial_l FH^i_{hj(s)} +$$
$$+ \dot\partial_l P_{(s)} Q^i_{hj} + \dot\partial_l PQ^i_{hj(s)} + \frac{2}{3} [\dot\partial^2_{l[h} P_{(s)} Q^i_{j]} + \dot\partial_m PG^m_{ls[h} Q^i_{j]} +$$
$$+ \dot\partial_l Q^i_{[j} \dot\partial_{h]} P_{(s)} + (\dot\partial_l Q^i_{[j\langle(s)\rangle} + Q^i_m G^m_{ls[j} - G^i_{mls} Q^m_{[j)}) \dot\partial_{h]} P +$$
$$+ 2 g_{l[h} H^i_{j](s)} + (\dot\partial_l H^i_{[j\langle(s)\rangle} + H^i_m G^m_{ls[j} - G^i_{mls} H^m_{[j)}) \dot\partial_{h]} F] ,$$

where we have made use of the following commutation formulae for any tensor $T^i_j(x, \dot{x})$

$$(2.16) \qquad (\dot{\partial}_h T)_{(k)} - \dot{\partial}_h T_{(k)} = 0 ,$$

$$(2.17) \qquad (\dot{\partial}^2_{lh} T)_{(k)} - \dot{\partial}^2_{lh} T_{(k)} = \dot{\partial}_m T G^m_{lhk}$$

and

$$(2.18) \qquad (\dot{\partial}_h T^i_j)_{(k)} - \dot{\partial}_k T^i_{j(k)} = T^i_m G^m_{hjk} - T^m_j G^i_{hmk}$$

and of the facts $F_{(h)} = 0$, $g_{lh(k)} = 0$.

Adding all the expressions obtained by cyclic permutation of the indices h, j and s and using the Bianchi identities (1.10) and (1.12), we get the results (2.12) and (2.13).

THEOREM (2.2). *The special projective tensor field* $M^i_{lhj}(x, \dot{x})$ *satisfies the following identities*:

$$(2.19) \qquad M^i_{[lhj]} = \dot{\partial}_{[l} FH_{hj]} + \dot{\partial}_{[l} PQ^i_{hj]} + \\ + \frac{1}{3} [\dot{\partial}_{[h} P\dot{\partial}_l Q^i_{j]} - \dot{\partial}_{[j} P\dot{\partial}_l Q^i_{h]} + \dot{\partial}_{[h} F\dot{\partial}_l H^i_{j]} - \dot{\partial}_{[j} F\dot{\partial}_l H^i_{h]}]$$

and

$$(2.20) \qquad M^i_{l[hj]} = 0 .$$

Proof. In view of equations (1.9 d), (1.10 c), (1.12 a) and (1.13 d), the equation (2.7) yields the theorem.

We define the expression for the special projective Veblen identity in Fn as follows:

$$(2.21) \qquad S^i_{lhjk}(x, \dot{x}) \overset{\text{def}}{=} \{M^i_{lhj(k)} + M^i_{jlk(h)} + M^i_{kjh(l)} + M^i_{hkl(j)}\} = 0 .$$

Thus we have

THEOREM (2.3). *In any Finsler space* Fn *the special projective Veblen identity is given by*

$$(2.22) \qquad S_{lphjk}(x, \dot{x}) = 0 .$$

Proof. Differentiating covariantly (2.7) with respect to x^k in the sense of Berwald and obtaining three more equations by interchanging the indices l, h, j, k and thus adding all four equations, we get

$$(2.23) \qquad M^i_{lhj(k)} + M^i_{jlk(h)} + M^i_{kjh(l)} + M^i_{hkl(j)} = B^i_{lhjk} + D^i_{lhjk} ,$$

where $B^i_{lhjk}(x, \dot{x})$ is the sum of the first 24 terms containing H^i_{lhj}, Q^i_{lhj}, H^i_{hj}, Q^i_{hj} and their derivatives and $D^i_{lhjk}(x, \dot{x})$ is defined the remaining last 48 terms.

Using equations (2.21) and (2.23), as a consequence of lowering of the indices we obtain

$$S_{lphjk} = B_{lphjk} + D_{lphjk} = 0 . \tag{2.24}$$

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