Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Bang-yen Chen, Huei-shyong Lue

# On complex submanifolds of complex projective space with constant scalar curvature

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **58** (1975), n.2, p. 172–173.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1975_8_58_2_172_0>

**Geometria differenziale.** — *On complex submanifolds of complex projective space with constant scalar curvature.* Nota di BANG-YEN CHEN (*) e HUEI-SHYONG LUE, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Per le varietà indicate nel titolo si dimostra che la curvatura normale scalare non è mai inferiore alla dimensione, l'uguaglianza avendosi se e soltanto se la varietà è localmente una sfeıa complessa.

## § 1. INTRODUCTION

Let $P_{n+1}(C)$ denote the $(n+1)$-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 1, and let $z_0, z_1, \cdots, z_{n+1}$ be a homogeneous coordinate system of $P_{n+1}(C)$. Let

$$Q_n = \{(z_0, z_1, \cdots, z_{n+1}) \in P_{n+1}(C) \mid \Sigma z_i^2 = 0\}.$$

Then, with respect to the induced Kaehler structure, $Q_n$ is an Einstein manifold with scalar curvature $n^2$ and scalar normal curvature $n$, and it is complex analytically isometric to the Hermitian symmetric space $SO(n+2)/SO(2)\times \times SO(n)$. We call $Q_n$ an $n$-dimensional complex sphere. In [4], Chen and Ogiue gave a characterization of $Q_n$ in terms of Ricci tensor. In this note, we shall prove the following

THEOREM. *Let* M *be an n-dimensional Kaehler submanifold immersed in* $P_m(C)$. *If the scalar curvature* $\rho$ *is equal to* $n^2$, *then the scalar normal curvature* $K_N$ *is* $\geqq n$, *and the equality holds if and only if* M *is locally* $Q_n$ *in some* $P_{n+1}(C)$ *in* $P_m(C)$.

## 2. BASIC FORMULAS

We prepare a brief summary of some basic facts. Details are found in [4] or [6].

Let M be an $n$-dimensional Kaehler submanifold immersed in $P_{n+p}(C)$. Let $\sigma$ or $A_\lambda$ be the second fundamental form of the immersion. Then the scalar curvature $\rho$ and the scalar normal curvature $K_N$ are given respectively by

$$\rho = n(n+1) - \|\sigma\|^2, \tag{1}$$

$$K_N = -\sum_{\lambda,\mu=1}^{2p} \operatorname{tr}(A_\lambda A_\mu - A_\mu A_\lambda)^2, \tag{2}$$

(*) Work done under partial support by NSF Grant GP–36684.
(**) Nella seduta dell'8 febbraio 1975.

where $\|\sigma\|$ denotes the length of $\sigma$. Moreover, $\sigma$ and $A_\lambda$ satisfy the following relations.

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\overline{\nabla}\sigma\|^2 - K_N - \Sigma(\operatorname{tr} A_\lambda A_\mu)^2 + \frac{n+2}{2}\|\sigma\|^2. \tag{3}$$

$$\Sigma(\operatorname{tr} A_\lambda A_\mu)^2 \leqq \|\sigma\|^4, \tag{4}$$

where $\overline{\nabla}$ denotes the van der Waeden connection (see, for instance, [1]). The equality in (4) holds if and only if rank $(\operatorname{tr} A_\lambda A_\mu) \leqq 2$. Geometrically speaking, the equality in (4) holds if and only if the dimension of the first normal space (the complex vector space spanned by the image of the second fundamental form) is at most 1.

## § 3. PROOF OF THE THEOREM

If the scalar curvature $\rho$ is equal to $n^2$, then $\|\sigma\|^2 = n$. Hence (3) implies

$$\|\overline{\nabla}\sigma\|^2 = K_N + \Sigma(\operatorname{tr} A_\lambda A_\mu)^2 - \frac{n+2}{2}\|\sigma\|^2 \leqq K_N - n\,. \tag{5}$$

From this we find $K_N \geqq n$. If we have $K_N = n$, then (5) gives $\|\overline{\nabla}\sigma\|^2 \leqq 0$, which implies

$$\overline{\nabla}\sigma = 0$$

and that the equality in (4) holds. Hence the first normal space is parallel and of dimension 1, which implies that M is immersed in some $P_{n+1}(C)$ in $P_{n+p}(C)$ (Thecrem 2.3 in [3]). On the other hand, $K_N = n = \|\sigma\|^2$ implies that M is Einstein. So, by a result of S.-S. Chern [5], M is locally $Q_n$ in $P_{n+1}(C)$.

## REFERENCES

[1] B.-Y. CHEN (1973) – *Geometry of Submanifolds*, Mercel Dekker, New York.
[2] B.-Y. CHEN and H.-S. LUE – *Differential geometry of* SO $(n+2)$/SO $(2)\times$SO $(n)$, Geometriae Dedicata (to appear).
[3] B.-Y. CHEN and K. OGIUE (1973) – *Some extrinsic results for Kaehler submanifolds*, « Tamkang J. Math. », *4*, 207–213.
[4] B.-Y. CHEN and K. OGIUE (1974) – *A characterization of complex spheres*, « Michigan Math. J. », *21*, 231–232.
[5] S.-S. CHERN (1967) – *Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature*, « J. Differential Geometry », *1*, 21–31.
[6] K. OGIUE (1974) – *Differential geometry of Kaehler submanifolds*, « Advances in Math. », *13*, 73–114.